ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ
W BYDGOSZCZY
Problemy Matematyczne 1982 z.4

Andrzej Nowicki
UMK Toruń

Ryszard Żuchowski
WSP Bydgoszcz

SOME REMARKS ON SYSTEMS OF IDEALS

If R is a commutative ring with identity and M is any set of ideals of R, then a pair (R, M) will be called a system of ideals, if the following conditions are satisfied

A1. R is an element of M,

A2. An intersection of any set of elements of M is an element of M,

A3. A union of any non-empty set, totally ordered by inclusion, of elements of M is an element of M,

A4. The null ideal belongs to M,

A5. If A, B belong to M, then A+B belongs to M,

A6. If A, B belong to M, then AB belongs to M,

A7. If A, B belong to M, then A:B belongs to M, where $A:B = \{r \in R; \bigwedge_{b \in B} r \cdot b \in A\}$

A8. If A belongs to M, and x is any element of R, then $A_x = \bigcup_{n=0}^{\infty} (A:x^n)$ belongs to M.

Let (R, M) be a system of ideals. Elements of M are called M-ideals. If E is a subset of R, then we denote by $[E]$ the smallest M-ideal containing E. If A is an ideal of R, then we denote by $A_{\#}$ the greatest M-ideal contained in A.

If (R, M) and (S, N) are systems of ideals, then a ring homomorphism $f:R \longrightarrow S$ will be called a morphism of systems if:

1. The inverse image of any N-ideal is an M-ideal,

2. An ideal generated in S by the image of any M-ideal is an N-ideal.

In the papers [7], [9], there are elements of the theory of systems of ideals and among others important examples of systems as i.e. systems of differential ideals ([4], [5]), systems of differential ideals with respect to a higher derivation ([2], [3]) and systems of homogenous ideals in a ring with a grading ([1], [8]).

In this paper we give some remarks on systems of ideals and we describe some new examples of those.

PROPOSITION 1. If M is a set of ideals of R satisfying A3, then the condition A8 is equivalent with the following condition:

(i) If $A \in M$, then $A_T \in M$, for every multiplicatively set T in R, where $A_T = \{ r \in R;\ rt \in A, \text{ for some } t \in T \}$.

PROOF. The property (i) implies A8, because $T = \{ x^n;\ n = 0, 1, 2, \ldots \}$ is a multiplicatively set in R and $A_x = A_T$. Now, we prove the inverse implication. Let $A \in M$ and let T be a multiplicatively set in R. Consider a family $\Sigma = \{ S;\ S$ is a multiplicatively set in R, such that $S \subset T$ and $A_S \in M \}$. The family Σ is non-empty, because for $t \in T$, $\{ t^n,\ n = 0, 1, 2, \ldots \}$ is a multiplicatively set contained in T, and by A8, we have $A_{\{t^n\}} = A_t \in M$. Notice, that Σ satisfies the assumption of Lemma Kuratowski-Zorn. Let $\{ S_i;\ i \in I \}$ be a chain in Σ. Then $\{ A_{S_i};\ i \in I \}$ is a chain in M and $\bar{S} = \bigcup_{i \in I} S_i$ is a multiplicatively set in R, contained in T. Since $A_{\bar{S}} = \bigcup_{i \in I} A_{S_i}$, then, by A3, we have $A_{\bar{S}} \in M$. Thus $\bar{S} \in \Sigma$.

Let $S \subset T$ be a maximal element in Σ. Thus $A_S \in M$. Suppose that $S \subsetneq T$, and let $t \in T \setminus S$. Then $U = S\{t^n\}$ is a multiplicatively set contained in T and properly containing S. By A3, we have $A_U = A_{S\{t^n\}} = (A_S)_t \in M$. Therefore $S \subsetneq U$ and $U \in \Sigma$, in spite of S is a maximal element in Σ. So, we get $S = T$, and $A_T \in M$.

PROPOSITION 2. If (R, M) is a system of ideals, then an algebraic sum of any set of elements of M, is an element of M.

PROOF. If A is an ideal of R, then the condition $A \in M$ is equivalent with the implication: $x \in A \rightarrow [x] \subset A$ (see [6]). Let $\{A_i\}_{i \in I}$ be a collection of M-ideals. If $x \in \sum A_i$ then $x = a_1 + \ldots + a_n$ belongs to $A_{i_1} + \ldots + A_{i_n}$, so $[x] \subset A_{i_1} + \ldots + A_{i_n} \subset \subset \sum A_i$.

Assume now, that (R, M) is a system of ideals, S is a multiplicatively set in R and $S^{-1}R$ is a quotient ring of R with respect to S. Let $N = \{ S^{-1}A;\ A \in M \}$. It is easy to prove, that N is the only set of ideals in $S^{-1}R$ such that $(S^{-1}R, N)$ is a system of ideals and the natural homomorphism $f: R \rightarrow S^{-1}R,\ r \mapsto \frac{r}{1}$ is a morphism of systems. The set N we shall denote by $S^{-1}M$.

PROPOSITION 3. If S is a multiplicatively set in R and $A \in I(R)$, then

a) $(S^{-1}A)_{\#} = S^{-1}((A_S)_{\#})$

b) $[S^{-1}A] = S^{-1}[A]$

PROOF. First, we prove, that if $A = A_S$, then $(S^{-1}A)_{\#} = S^{-1}A_{\#}$.

It is clear, that $(S^{-1}A)_{\#} = S^{-1}B$, where $B \in M$ and $B = B_S$.
Hence $A_{\#} = (A_S)_{\#} = (f^{-1}(S^{-1}A))_{\#} = f^{-1}((S^{-1}A)_{\#}) = f^{-1}(S^{-1}B)$
$= B_S = B$,
where $f: R \rightarrow S^{-1}R$ is the natural homomorphism.
Then we have $S^{-1}(A_{\#}) = S^{-1}B = (S^{-1}A)_{\#}$, so finally

$$(S^{-1}A)_{\#} = (S^{-1}(A_S))_{\#} = S^{-1}((A_S)_{\#}).$$

This ends the proof of a). The proof of b) is standard.

A system (R, M) is called special, if the radical of an arbitary M-ideal is an M-ideal ([7]).

PROPOSITION 4. If S is a multiplicativaly set in R, and (R,M) is a special system, then $(S^{-1}R, S^{-1}M)$ is special too.

PROOF. Let Q be any prime ideal in $S^{-1}R$. Then $Q = S^{-1}P$, where P is a prime ideal in R, disjoint from S. By proposition 3 a), we have $Q_{\#} = (S^{-1}P)_{\#} = S^{-1}(P_{\#})$. But (R, M) is special, so $P_{\#}$ is a prime ideal of R (Th.1.2 [7]). Since $P_{\#} \subset P$ and $P \cap S = \emptyset$ then $P_{\#} \cap S = \emptyset$ Finally, $Q_{\#} = S^{-1}P_{\#}$ is a prime ideal in $S^{-1}R$ and by [7] we have thesis.

If P is a prime ideal in R, then we denote by (R_P, M_P) a system $(S^{-1}R, S^{-1}M)$, where $S = R \setminus P$.

PROPOSITION 5. Let (R, M) be a system of ideals. The following conditions are equivalent

(1) (R, M) is special.

(2) (R_P, M_P) is special, for every prime ideal P in R.

(3) (R_{M_1}, M_{M_1}) is special, for every maximal ideal M_1 in R.

PROOF. The implication $(1) \Rightarrow (2)$ follows by Proposition 4. It is clear, that $(2) \Rightarrow (3)$. We prove $(3) \Rightarrow (1)$.
Consider any prime ideal P in R. We show, that $P_{\#}$ is a prime ideal. Let M_1 be a maximal ideal, such that $P \subset M_1$ Let $S = R \setminus M_1$. Then $S \cap P = \emptyset$, and by Proposition 3 we have $(S^{-1}P)_{\#} = S^{-1}(P_{\#})$
Because $(S^{-1}R, S^{-1}M)$ is special, so $(S^{-1}P)_{\#}$ is a prime ideal in $S^{-1}R$, hence $P_{\#}$ is a prime ideal in R.

Now we give new examples of systems of ideals. First, we describe all systems of ideals in the ring Z of integers.

EXAMPLE 1. Let $P = \{p_1, p_2, \ldots\ldots, p_n, \ldots\}$ be a set of prime integers (finite or infinite), and let $D = \{p_1^{i_1}, p_2^{i_2}, \ldots, p_n^{i_n}, \ldots\}$ be a set of fixed powers of elements of P. Then (Z, M_D) where $M_D = \{(n);\ n = (p_1^{i_1})^{s_1} \ldots (p_k^{i_k})^{s_k},\ s_i \geqslant 0\} \cup \{(0)\}$ is a system of ideals. Conversely, every system (Z, M) has the above form.

PROOF. It is easy to prove, that (Z, M_D) is a system of ideals. We show, that any system of ideals (Z, M) has the form (Z, M_D). Let $P = \{p;\ p$ - a prime integer, such that $p | n$ for some $(n) \in M, n \neq 0\}$

If $P = \{p_1, p_2, \ldots, p_k, \ldots\}$, then we define
$i_j = \min\{i;$ there exists $(n) \in M$: $n = p_j^i c$, $i > 0$, $p_j \nmid c\}$, where
$j = 1,2,\ldots$ Set $D = \{p_1^{i_1}, p_2^{i_2}, \ldots, p_k^{i_k}, \ldots\}$. We prove that $M = M_D$.
Notice, that if $(n) \in M$, where $n = p_1^{k_1} \cdot \ldots \cdot p_s^{k_s}$, then by the uniqueness of primary decomposition and by Th. 3.4 [8], we obtain $(n) = (p_1^{k_1}) \cap \ldots \cap (p_s^{k_s})$, where $(p_1^{k_1}) \in M, \ldots, (p_s^{k_s}) \in M$. Hence, also $(p_1^{i_1}) \in M, \ldots, (p_j^{i_j}) \in M, \ldots$

Therefore $M_D \subset M$. If $(m) \in M$, where $m = p_1^{k_1} \cdot \ldots \cdot p_t^{k_t}$, then $(p_j^{k_j}) \in M$, for $j = 1, 2, \ldots, t$. It is obvious that $k_j \geq i_j$.

Let $k_j = u_j i_j + r_j$, where $0 \leq r_j \leq i_j$, $j = 1, 2, \ldots, t$. If $r_j \neq 0$, then $(p_j^{r_j}) = (p_j^{k_j}) : (p_j^{u_j i_j}) \in M$, which contradicts with minimality of i_j. Hence, we have $(p_j^{k_j}) = (p_j^{i_j})^{u_j} \in M_D$ for $j = 1, 2, \ldots, t$.
Finally $(m) = (p_1^{k_1}) \cap \ldots \cap (p_t^{k_t}) \in M_D$ and $M = M_D$

We shall describe now, all special systems in Z.

EXAMPLE 2. Let P be a set of prime integers. Then (Z, M_P), where $M_P = \{(n);\ n = p_1^{i_1} \cdot \ldots \cdot p_k^{i_k};\ p_1, p_2, \ldots \in P,\ i_1 \geq 0, \ldots, i_k \geq 0\} \cup \{(0)\}$ is a special system. Conversely, if (Z, M) is a special system, then there exists a set P of prime integers such that $M = M_P$.

PROOF. Is similar to the proof of Example 1.

We can do the analogous description for principal ideals domains.

If (R, M) is a system of ideals in R, then we denote by $M[x]$ a set of ideals in $R[x]$ of the form $A[x] = \{a_0 x^n + \ldots + a_n;\ a_i \in A\}$, where $A \in M$. We shall, prove, that $(R[x], M[x])$ is a system of ideals. First, we prove two lemmas.

LEMMA 1. Let $f = a_n x^n + \ldots + a_0$, $g = b_m x^m + \ldots + b_0$ belong to $R[x]$. If $g \cdot f = 0$, then $g \cdot a_n^{m+1} = 0$

PROOF. Since $g.f = 0$, we have

(1) $b_m \cdot a_n = 0$

(2) $b_{m-1}a_n + b_m a_{n-1} = 0$

. . . .

(m) $b_1 a_n + b_2 a_{n-1} + \ldots\ldots + b_{n+1} a_o = 0$

(m+1) $b_o a_n + b_1 a_{n-1} + \ldots\ldots + b_n a_o = 0$

Multiply the equation (2) by a_n, (3) by a_n^2, ... , (m+1) by a_n^m, we have: $b_{m-1}a_n^2 = 0$,

$b_{m-2}a_n^3 = 0$,

. . . .

$b_1 a_n^m = 0$,

$b_O a_n^{m+1} = 0$

Hence $g.a_n^{m+1} = 0$.

LEMMA 2. Let $f = a_n x^n + \ldots + a_O$, $g = b_m x^m + \ldots + b_0$ belong to $R[x]$. . If $g \in \bigcup_{k=0}^{\infty} (0:f^k)$, then for each pair i, j, where $i = 0, 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$ there exists $s(i, j) \in N$ such that $b_i \cdot a_j^{s(i, j)} = 0$.

PROOF. Induction with regard of n, where $n = \deg f$. If $n = 0$, the Lemma is obvious. Suppose now, that Lemma is true for polynomials f of degrees $< n$, and for every polynomial g. Let $\deg f = n$, and $g \in \bigcup_{k=o}^{\infty} (0:f^k)$.

Suppose, that $g.f^p = 0$. Let $h = f^p$. A coefficient at the maximal power of x in h is equal: $a_n^p = b$ Since $g.h = 0$, by the Lemme 1, we have $g.b^{m+1} = 0$. Then $g \cdot a_n^r = 0$, where $r = p(m+1)$.

Let $f_1 = f - a_n x^n$, then $g \cdot f_1^{p+r} = 0$. Finally $g \in \bigcup_{k=o}^{\infty} (0:f_1^k)$, where $\deg f_1 \leqslant n-1$, the thesis follows by the induction assumption.

Corollary. Let A be an ideal in R and let $f = a_n x^n + \dots + a_o$, $g = b_m x^m + \dots + b_o$ be polynomials in $R[x]$. If $g \in \bigcup_{k=o}^{\infty} (A[x] : f^k)$, then for every pair i,j, where $i = 0,1,2, \dots, m$, $j = 1, 2, \dots, n$, there exists $s(i, j) \in N$ such that $b_1 \cdot a_j^{s(i, j)} \in A$.

PROOF. It suffices to apply Lemma 2 and the isomorphism $(R/A)[x] = R[x]/A[x]$

THEOREM. Let $f = a_n x^n + \dots + a_o \in R[x]$ and let A be and ideal of R. Then $A[x]_f = (A_{a_o} \cap A_{a_1} \cap \dots A_{a_n})[x]$

i.e. $\bigcup_{k=o}^{\infty} (A[x] : f^k) = \bigcap_{i=o}^{n} \left(\bigcup_{k=o}^{\infty} (A:a_1^k) \right) [x]$ . .

PROOF. The inclusion $\subset$ follows from Corollary, we prove the inverse inclusion. If $g = b_m x^m + \dots + b_o$ belongs to $\bigcap_{i=o}^{n} \left(\bigcup_{k=o}^{\infty} (A:a_i^k) \right) [x]$, then $b_k a_o^{s_{ok}}, \dots, b_k a_n^{s_{nk}} \in A$, Set $s_k = \max(s_{ok}, \dots, s_{nk})$, for $k = 1, 2, \dots m$. Then $b_k a_i^{s_k} \in A$ and $g \cdot f^{s_0 + s_1 + \dots + s_m} \in A[x]$.

EXAMPLE 3. If (R, M) is a systems of ideals, then (R[x],M[x]) is a system of ideals too.

PROOF. It is clear, that the conditions A1-A7 are satisfied. By the obove Theorem, the condition A8 is also satisfied.

EXAMPLE 4. If (R, M) is a special system of ideals, then (R[x], M[x]) is a special system of ideals.

PROOF. It suffices to notice, that $f(A[x]) = r(A)[x]$.

REFERENCES

[1] M.F.Atiyah, I.G.Macdonald, Introduction to Commutative Algebra Adison - Weseley Publishing Company, Massachusetts 1969

[2] W.C.Brown, W.Kuan, Ideals and higher derivation in commutative rings, Canadian Journal of Mathematics 24(1972) . .

[3] N.Jacobson, Lectures in Abstract Algebra vol. I I I, D. Van Nostrand Company. 1964

[4] W.F.Keigher, Prime differential ideals in diferential rings, Contribution to Algebra, A Collection of Papers Dedicated to E.Kolchin, 1977, 239-249.

[5] E.R.Kolchin, Differential Algebra and Algebraic Groups, Academic Press, New York, London, 1973.

[6] A.Nowicki, Prime ideals structure in additiwe conservative systems (to appear) . .

[7] A.Nowicki, R.Żuchowski, Special systems of ideals in commutative rings (to appear in. Commentationes Mathematicae).

[8] O.Zariski, P.Samuel, Commutative Algebra vol. 2, Van Norstrand C.O, Princeton. 1960

[9] R.Żuchowski, Systems of ideals in commutative rings (to appear in Commentationes Mathematicae).

PEWNE UWAGI O SYSTEMACH IDEAŁÓW

STRESZCZENIE

W pracach [7], [9] podane są elementy teorii systemów ideałów i między innymi ważniejsze przykłady systemów jak np. systemy ideałów różniczkowych, systemy ideałów niezmienniczych ze względu na derywację wyższą i systemy ideałów jednorodnych w pierścieniach z gradacją.

W niniejszej pracy podane są pewne uwagi dotyczące systemów i opisane są nowe przykłady systemów ideałów.