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TADEUSZ MARIAN JĘDRYKA
Bydgoszcz

AN ESTIMATION OF THE SOLUTION OF VOLTERRA'S INTEGRAL EQUATION FOR VECTOR-VALUED FUNCTIONS WITH VALUES IN SOME FUNCTION SPACES

In [2] we dealt with a Volterra integral equation

$$(1) \qquad u(x) = \int_a^x T(x,t)u(t)dt + b(x) ,$$

where $x,t,a \in R^n$; b and u are vector-valued functions of the variable t $(a \leq t \leq x)$ with values in a Banach space Y, and $T(x,t)$ is a linear bounded operator of Y into itself for $a \leq t \leq x$, strongly measurable in both variables.

The order relation $c \leq d$ for $c = (c_1,\dots,c_n) \in R^n$, $d = (d_1,\dots,d_n) \in R^n$ means here that $c_i \leq d_i$ for $i = 1,2,\dots,n$. Denoting by $\mathcal{Y}$ the space of all such operators and supposing that $\|b(x)\|_Y \leq B(x)$ where $B(x)$ is measurable and bounded for $x \geq a$, there was proved the following theorem :

Theorem. Let us suppose that $A(x,t)$ is a real-valued function, defined for $a \leq t \leq x$, nondecreasing with respect to x for every t and such that $A(t,t)$ is measurable and bounded for $t \geq a$; $\alpha \int_a^x A(t,t)dt < 1$ for $x \geq a$. Moreover, let us suppose that

$$(2) \qquad \|T(x,t)\|_{\mathcal{Y}} \leq \alpha A(x,t) \quad \text{for } a \leq t \leq x ,$$

where α is independent of x and t. Then the integral equation (1) has a unique solutions in the space of all Y-valued bounded and strongly measurable functions in $x \geq a$. Moreover we have an estimation

$$\|u(x)\|_Y \leq \beta(x)\exp\left(\int_a^x \alpha A(t,t)\,dt\right) \quad \text{for } x \geq a,$$

where $\beta(x) = \sup\limits_{a \leq t \leq x} B(x)$, and where $\|b(x)\|_Y \leq B(x)$.

1. The aim of this section is to verify the inequality (2) in the case when the operator $T(x,t)$, which will be briefly denoted by T, is given by the equality $v = Tu$, where $v_\sigma = \int_a^b T_{\sigma,\tau}\,du_\tau$ and Y is the space CV_p of continuous functions u of bounded p-variation $V_p(u)$ in $\langle a,b\rangle$, $p > 1$. Let us recall that the p-variation of u in $\langle a,b\rangle$ is defined by

$$V_p(u) = \sup_{\Pi} \sum_{i=1}^{n} \left|u(\tau_i) - u(\tau_{i-1})\right|^p,$$

where Π: $a = \tau_0 < \tau_1 < \ldots < \tau_n = b$ is an arbitrary division of the interval $\langle a,b\rangle$ (see L.C.Young, [5]), V_p will mean the space of all functions u for which $V_p(u) < +\infty$. Then $\|u\|_p = \left(V_p(u)\right)^{1/p}$ is a seminorm in both, in CV_p and V_p and $\|u\|_p = 0$ if and only if u is constant in $\langle a,b\rangle$. L.C.Young proved, [5], that if $p,q > 1$, $\frac{1}{p} + \frac{1}{q} > 1$ and $u \in CV_p$, $w \in V_q$, whan the Riemann-Stieltjes integral $\int_a^b w\,du$ exists and

$$\left|\int_a^b w\,du\right| \leq K\|w\|_q\|u\|_p\,\zeta\left(\frac{1}{p} + \frac{1}{q}\right) \quad \text{where } \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

We shall prove now the following theorem:

Theorem 1. Let $p,q > 1$, $\frac{1}{p} + \frac{1}{q} > 1$ and let a real-valued function $T_{\sigma,\tau}$ defined for $\sigma,\tau \in \langle a,b\rangle$ satisfy the following conditions:

(a) $T_{\sigma,\cdot} \in V_q$ and $T_{\sigma,a} = 0$ for every $\sigma \in \langle a,b\rangle$, $T_{a,\cdot} = 0$,

(b) the vector-valued function $f: \langle a,b\rangle \to V_q$ defined by $f(\sigma) = T_{\sigma,\cdot}$ is continuous in $\langle a,b\rangle$,

(c) $A = \left\{\sup\limits_{\Pi} \sum_{i=1}^{n} \|T_{\sigma_i,\cdot} - T_{\sigma_{i-1},\cdot}\|_q^p\right\}^{\frac{1}{p}} < +\infty$.

Then the operator T defined by $v = Tu$ with

$$v_\sigma = \int_a^b T_{\sigma,\tau}\, du_\tau$$

is a linear continuous operator in CV_p and its norm

$$\|T\|_{\gamma} \leqslant \alpha A$$

with

$$\alpha = K\,\xi\left(\frac{1}{p} + \frac{1}{q}\right).$$

Proof. From the condition (b) it follows that $v = v_\sigma$ is a continuous function in $\langle a,b\rangle$, because for arbitrary $\sigma_1, \sigma_2 \in \langle a,b\rangle$ we have

$$\left|v_{\sigma_1} - v_{\sigma_2}\right| = \left|\int_a^b (T_{\sigma_1,\tau} - T_{\sigma_2,\tau})du_\tau\right| \leqslant$$

$$\leqslant K\|T_{\sigma_1,\cdot} - T_{\sigma_2,\cdot}\|_q \|u\|_p\, \xi\left(\frac{1}{p} + \frac{1}{q}\right).$$

Now, we have

$$\|v\|_p = \left\{\sup_{\Pi} \sum_{i=1}^{n} \left|\int_a^b \left(T_{\sigma_i,\tau} - T_{\sigma_{i-1},\tau}\right) du_\tau\right|^p\right\}^{\frac{1}{p}} \leqslant$$

$$\leq \left\{\sup_{\Pi} \sum_{i=1}^{n} K\|T_{\sigma_i,\cdot} - T_{\sigma_{i-1},\cdot}\|_q^p \|u\|_p^p\, \xi^p\left(\frac{1}{p} + \frac{1}{q}\right)\right\}^{\frac{1}{p}} =$$

$$= KA\,\xi\left(\frac{1}{p} + \frac{1}{q}\right)\|u\|_p.$$

This completes the proof.

2. Now, keeping the operator T in the form $v = Tu$ with $v_\sigma = \int_a^b T_{\sigma\tau}\, du_\tau$ let us consider the case if $Y = V_1$ where V_1 is the space of functions of bounded variation $\|u\|_1 = V_1(u)$ in $\langle a,b\rangle$. The following theorem holds:

Theorem 2. Let $T_{\sigma,\tau}$ be a real-valued function defined for $\sigma,\tau \in \langle a,b\rangle$ and satisfying the following conditions:

(a) $T_{\sigma,\tau}$ is continuous with respect to τ for every $\sigma \in \langle a,b\rangle$,

(b) $T_{\sigma,\tau}$ satisfies the Lipschitz condition with respect to σ for every $\tau \in \langle a,b \rangle$:

$$|T_{\sigma',\tau} - T_{\sigma'',\tau}| \leq A|\sigma' - \sigma''|$$

with the constant A independent of τ.

Then the operator T defined as in Theorem 1 is a continuous linear operator in V_1 and its norm $\|T\|_{\mathcal{Y}} \leq \alpha A$ with $\alpha = b - a$.

Proof. We have

$$\|v\|_1' = \sup_{\Pi} \sum_{i=1}^{n} \Big| \int_a^b \left(T_{\sigma_i,\tau} - T_{\sigma_{i-1},\tau} \right) du_\tau \Big| \leq A(b - a)\|u\|_1 .$$

3. Finally, we are going to estimate the norm $\|T\|_{\mathcal{Y}}$ in case of the space Y = H of functions satisfying the Hölder condition with an exponent γ, $0 < \gamma \leq 1$, in the interval $\langle a,b \rangle$, the norm in H is defined by

$$\|u\|_H = |u(a)| + \sup_{\tau',\tau'' \in \langle a,b \rangle} \frac{|u_{\tau'} - u_{\tau''}|}{|\tau' - \tau''|^{\gamma}}$$

There holds

Theorem 3. Let $T_{\sigma\tau}$ be a real-valued function defined for $\sigma,\tau \in \langle a,b \rangle$ and such that

(a) $T_{\sigma,\tau}$ is an intergrable function of τ in $\langle a,b \rangle$ for every $\sigma \in \langle a,b \rangle$

(b) $T_{\sigma,\tau}$ satisfies the Hölder condition

$$|T_{\sigma',\tau} - T_{\sigma'',\tau}| \leq A|\sigma' - \sigma''|^{\gamma}$$

for every $\tau \in \langle a,b \rangle$, with A independent of τ.

Then the operator T defined as $v = Tu$ with $v_\sigma = \int_a^b T_{\sigma\tau} u_\tau d\tau$ is a continuous linear operator in H with norm

$$\|T\|_{\mathcal{Y}} \leq \alpha A$$

where

$$\alpha = (b - a)\max\left(1, \frac{(b-a)^{\gamma}}{\gamma+1}\right).$$

Proof. We obtain

$$\|v\|_H = \sup_{\substack{\sigma',\sigma''\in\langle a,b\rangle \\ \sigma'\neq\sigma''}} \frac{\left|\int_a^b \left(T_{\sigma',\tau} - T_{\sigma'',\tau}\right) u_\tau \, d\tau\right|}{|\sigma' - \sigma''|^\gamma} \leq$$

$$\leq \sup_{\substack{\sigma',\sigma''\in\langle a,b\rangle \\ \sigma'\neq\sigma''}} \int_a^b \frac{\left|T_{\sigma',\tau} - T_{\sigma'',\tau}\right|}{|\sigma' - \sigma''|^\gamma} |u_\tau| \, d\tau \leq A \int_a^b |u_\tau| \, d\tau .$$

But

$$|u_\tau| \leq |u(a)| \left[1 - (\tau - a)^\gamma\right] + \|u\|_H (\tau - a)^\gamma$$

for every $\tau \in \langle a, b\rangle$.

Hence

$$\|v\|_H \leq A(b-a)\left[|u(a)|\left(1 - \frac{(b-a)^\gamma}{\gamma+1}\right) + \|u\|_H \frac{(b-a)^\gamma}{\gamma+1}\right] \leq$$

$$\leq A(b-a)\max\left(1, \frac{(b-a)^\gamma}{\gamma+1}\right)\|u\|_H .$$

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AN ESTIMATION OF THE SOLUTION OF VOTERRA'S INTEGRAL EQUATION FOR VECTOR-VALUED FUNCTIONS WITH VALUES IN SOME FUNCTION SPACES

SUMMARY

There are given proofs of three theorems in this paper, in which is defined the constante of the estimation of the solution of Volterra's integral equation when the values of the kernel are in the CV_p and V_1 spaces and also in case when the kernel satisfy the Hölder condition regard to parameter.