ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ W BYDGOSZCZY
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AN ESTIMATION OF THE SOLUTION OF VOLTERRA'S INTEGRAL EQUATION FOR VECTOR-VALUED FUNCTIONS WITH VALUES IN AN ORLICZ SPACE

1. In [2] we dealt with a Volterra integral equation

$$(1)\quad u(x) = \int_a^x T(x, t)\, u(t)\, dt + b(x)$$

where $x, t, a \in R^n$; b and u are vector - valued functions of the variable $t (a \leq t \leq x)$ with values in a Banach space Y, and $T(x, t)$ is a linear bounded operator of Y into itself for $a \leq t \leq x$, strongly measurable in both variables.

The order relation $c \leq d$ for $c = (c_1, \ldots, c_n) \in R^n$, $d = (d_1, \ldots, d_n) \in R^n$ means here that $c_i \leq d_i$ for $i = 1, 2, \ldots, n$. Denoting by $\mathcal{Y}$ the space of all such operators and supposing that $\|b(x)\|_Y \leq B(x)$ where $B(x)$ is measurable and bounded for $x \geq a$, there was proved the following theorem:

Let us suppose that $A(x, t)$ is a realvalued funtion, defined for $a \leq t \leq x$, nondecreasing with respect to x for every t and such that $A(t, t)$ is measurable and bounded for $t \geq a$, $\alpha \int_a^x A(t, t)\, dt < 1$ for $x \geq a$. Moreover, let us suppose that.

$$(2)\quad \|T(x, t)\|_{\mathcal{Y}} \leq \alpha A(x, t) \text{ for } a \leq t \leq x,$$

where α is independent of x and t.

Then the integral equation (1) has a unique solu-

tion in the space of Y - valued bounded and strongly measurable functions in $x \geqslant a$.
Moreover, we have an estimation

$$\|u(x)\|_Y \leqslant \beta(x) \exp\left(\int_a^x \alpha\, A(t,t)\, dt\right) \text{ for } x \geqslant a,$$

where $\beta(x) = \sup_{a \leqslant t \leqslant x} B(t)$.

2. In this paper there will be verified the inequality (2) in case when Y is anOrlicz space L^φ. Let Ω be a σ-algebra of subsets of a nonempty set E and let μ be a non-trivial measure in E. Moreower let $\varphi(s)$ be a function defined in $\langle 0, \infty)$, satisfying the following conditions:

(a) $\varphi(s) \geqslant 0$; $\varphi(s) = 0$ iff $s = 0$.
(b) $\varphi(s)$ is an even, continuous and convex function.

Then the set of all measurable, extended real - valued functions u on E such that $\int_E \varphi(\lambda|u(t)|)\mu(dt) < \infty$ for some $\lambda > 0$, is a Banach space. This space is defined as the space L^φ generated by means of the function φ, (see [4]).

Let φ^* be function conjugate to φ in the sense of Young, i.e.

$$\varphi^*(s) = \sup_{\sigma \geqslant 0} (s\sigma - \varphi(\sigma)) \quad \text{for } s \geqslant 0$$

$$\varphi^*(s) = \varphi^*(|s|) \text{ for } s < 0.$$

Let us write

$$\rho(u) = \int_E \varphi(|u(\tau)|)\mu(d\tau) \; ; \quad \rho^*(u) = \int_E \varphi^*(|u(\tau)|)\mu(dt)$$

The following two norms are defined in L^φ:

$$\|u\|_\varphi = \inf\left\{\delta > 0 : \rho\left(\frac{u}{\delta}\right) \leq 1\right\} \text{ - Luxemburg norm,}$$

$$\|u\|_\varphi^0 = \sup\left\{\int_E u(\tau)\, w(\tau)\, \mu(d\tau) : \rho^*(w) \leq 1\right\} -$$

- Orlicz norm;

we have $\|u\|_\varphi \leq \|u\|_\varphi^0$ (in case of φ dependent only on s, see : [3]). Also the unit ball $\{u \in L^\varphi : \|u\|_\varphi \leq 1\}$ is equal to the set $\{u \in L^\varphi : \varrho(u) \leq 1\}$ (see : [3], theorem 9.5, p. 96).

3. Let T be defined by the formula $v = Tu$ for $u \in L^\varphi$, where $v_\sigma = \int_E T_{\sigma,\tau} \; u_\tau \; \mu(d\tau)$, $\tau \in E$, $\sigma \in E$.

Then we have, taking $\mathcal{Y}$ with the norm, generated by means of the Luxemburg norm in L^φ :

$$\|T\|_{\mathcal{Y}} = \sup_{\|u\|_\varphi \leq 1} \|Tu\|_\varphi = \sup_{\varrho(u) \leq 1} \|Tu\|_\varphi = \sup_{\varrho(u) \leq 1} \left\|\int_E T_{\sigma,\tau} u_\tau \mu(d\tau)\right\|_\varphi \leq$$
$$\leq \sup_{\varrho(u) \leq 1} \left\|\int_E T_{\sigma,\tau} u_\tau \mu(d\tau)\right\|_\varphi^0 \leq$$
$$\leq \sup_{\varrho^*(w) \leq 1} \sup_{\varrho(u) \leq 1} \int_E \left\{\int_E |T_{\sigma,\tau} w_\sigma| \mu(d\sigma)\right\} |u_\tau| \mu(d\tau) =$$
$$= \sup_{\varrho^*(w) \leq 1} \left\|\int_E |T_{\sigma,\tau} w_\sigma| \mu(d\sigma)\right\|_{\varphi^*}^0.$$

Now, we apply this inequality to $T(x, t)$ in place of T. We suppose that there holds the inequality

(3) $|T_{\sigma,\tau}(x, t)| \leq A(x, t)$ for all x, t, σ, τ.
We suppose now that μ is a finite measure i.e. $\mu(E) < \infty$
Thus we obtain

$$\|T(x, t)\|_{\mathcal{Y}} \leq \sup_{\varrho^*(w) \leq 1} \left\|\int_E A(x, t) | w_\sigma| \mu(d\sigma)\right\|_{\varphi^*}^0 \leq$$
$$\leq \|1\|_{\varphi^*}^0 \sup_{\varrho^*(w) \leq 1} \int_E | w_\sigma| \mu(d\sigma) \cdot A(x, t).$$

By Jensen's inequality for convex function we have:

$$(4) \qquad \int_E |w_\sigma| \mu(d\sigma) \leq \mu(E)\, \varphi^{-1}\left(\frac{1}{\mu(E)} \varrho(w)\right)$$

$$(5) \qquad \int_E |w_\sigma| \mu(d\sigma) \leq \mu(E) (\varphi^*)^{-1}\left(\frac{1}{\mu(E)} \varrho^*(w)\right).$$

Hence, applying (4) we get

$$\|1\|_{\varphi^*}^0 = \sup_{\varrho(w) \leq 1} \int_E 1 . |w_\sigma| \mu(d\sigma) \leq \mu(E)\, \varphi^{-1}\left(\frac{1}{\mu(E)}\right) \quad ,$$

and aplying (5) we obtain

$$\sup_{\varphi^*(w)\leq 1} \int_E |w_\sigma| \mu(d\sigma) \leq \mu(E)(\varphi^*)^{-1}\left(\frac{1}{\mu(E)}\right).$$

Consequently

$$\|T(x,t)\|_{y} \leq (\mu(E))^2 \varphi^{-1}\left(\frac{1}{\mu(E)}\right)(\varphi^*)^{-1}\left(\frac{1}{\mu(E)}\right)\cdot A(x,t)$$

Thus, we obtained the following result:

4. If μ is finite $T(x, t)$ is defined as above and if there holds the inequality (3) then

(6) $\|T(x, t)\|_{y} \leq \alpha A(x, t)$

where

$$\alpha = (\mu(E))^2 \varphi^{-1}\left(\frac{1}{\mu(E)}\right)(\varphi^*)^{-1}\left(\frac{1}{\mu(E)}\right).$$

5. Let us observe that in case of $\varphi(u) = |u|^p$ with $p > 1$ i.e. $L^\varphi = L^p$, we have $\varphi^{-1}(u) = |u|^{1/p}$,

$\varphi^*(u) = |u|^q$, $(\varphi^*)^{-1}(u) = |u|^{1/q}$ where $\frac{1}{p} + \frac{1}{q} = 1$.

Hence we get

$$\alpha = \mu(E)$$

which is the result obtained in [2].

6. We omit now the assumption $\mu(E) < \infty$. Let two functions $e \in L^\varphi$ and $\varepsilon \in L^{\varphi^*}$ be given and let us suppose that in place of (3) there holds

(7) $|T_{\sigma,\tau}(x, t)| \leq e(\sigma)\varepsilon(\tau) A(x, t)$ for all x, t, σ, τ.

Then it is easily seen that

$$\|T(x, t)\|_{y} \leq \sup_{\varrho(w^*)\leq 1} \left\|\int_E e(\sigma)\,\varepsilon(\tau) A(x, t)\, |w_\sigma| \mu(d\sigma)\right\|^0_{\varphi^*} \leq$$

$$\leq \|\varepsilon\|^0_{\varphi^*} \sup_{\varrho(w^*)\leq 1} \int_E e(\sigma)\, |w_\sigma| \mu(d\sigma) \cdot A(x, t) =$$

$$= \|\varepsilon\|^0_{\varphi^*} \|e\|^0_{\varphi}\; A(x, t)$$

and we obtain the inequality (6) with

$$\alpha = \|e\|_{\varphi}^{0} \|\varepsilon\|_{\varphi^*}^{0} .$$

This includes the case considered in [2], 4, with φ as in 5 and $e(\sigma) = \frac{1}{\sigma}$; $\varepsilon(\tau) = \frac{1}{\tau}$.

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THERE IS INVESTIGATED A VOLTERRA INTEGRAL EQUATION

Summary

$$(1)\quad u(x) = \int_a^x T(x,\ t)\, u(t)\, dt + b(x)$$

where x, t, $a \in R^n$; b and u are vector - valued functions of the variable $t(a \leq t \leq x)$ with values in an Orlicz space L^{φ}, and $T(x,\ t)$ is a linear bounded operator of L^{φ} into itself for $a \leq t \leq x$.

The problem originates in some results of T. Ważewski concernig a differential equation and investigated by T. Sato and Z. Butlewski for systems of Volterra equations with real - valued functions.

In the paper [2] the autor of this note generalised the above result to the case of vector - valued functions among other with values in the space L^p. New, estimations are obtained in case of an Orlicz space L^{φ}.

O OSZACOWANIU ROZWIĄZANIA RÓWNANIA CAŁKOWEGO VOLTERRY DLA FUNKCJI WEKTOROWYCH O WARTOŚCIACH Z PRZESTRZENI ORLICZA

Streszczenie

Rozważa się równanie całkowe Volterry

$$u(x) = \int_a^x T(x,t)u(t)dt + b(x) , \tag{1}$$

gdzie $x,t,a \in R^n$; b oraz u są funkcjami wektorowymi zmiennej t ($a \leqslant t \leqslant x$) , o wartościach z przestrzeni Orlicza L^{φ} , zaś $T(x,t)$ jest liniowym, ograniczonym operatorem , działającym z przestrzeni L^{φ} w nią samą, dla $a \leqslant t \leqslant x$.

Zagadnienie to, zapoczątkowane przez T. Ważewskiego dla równania różniczkowego, było badane przez T. Sato i Z. Butlewskiego dla układu równań Volterry z funkcjami o wartościach rzeczywistych.

W pracy [2], autor tego artykułu uogólnił powyższe rezultaty na przypadek funkcji wektorowych, między innymi o wartościach w przestrzeni L^p .

Tutaj otrzymane oszacowanie dotyczy przypadku przestrzeni Orlicza L^{φ} .