ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ W BYDGOSZCZY
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SOME THEOREMS ON GENERALIZED MODULAR SPACES

1. In [4] there was introduced the notion of the modular space X_{ϱ_s} by means of a family of modulars depending on a parameter. This notion was applied in [1] and [2] to investigation of modular equations and integral equations of a special type. In this paper we investigate two problems in the space X_{ϱ_s} : this of density and separability and that of uniform continuity of a translation operator.

Let (Ω, Σ, μ) be a measure space, Σ being a σ- algebra of subsets of a nonempty set Ω and μ - a σ - finite, complete, positive measure on Σ. Let $\mathcal{X}'$ be the space of Σ - measurable, extended real-valued functions on Ω with equality μ - a.e. Let $\varrho : \Omega \times \mathcal{X} \to \langle 0, \infty)$ satisfy the following conditions:

1° $\varrho(t,x)$ is a pseudomodular over $\mathcal{X}$ (for definition, see [3]) for a.e. $t \in \Omega$,

2° if $\varrho(t,x) = 0$ for a.e. $t \in \Omega$, then $x = 0$,

3° $\varrho(t,x)$ is Σ - measurable in Ω for every $x \in \mathcal{X}$,

4° if $x, y \in \mathcal{X}$ and $|x(t)| \leqslant |y(t)|$ a.e. in Ω, then $\varrho(t,x) \leqslant \varrho(t,y)$ a.e. in Ω.

Let X be the set of $x \in \mathcal{X}$ for which $\varrho(t, \lambda x) \to 0$ as $\lambda \to 0$ a.e. in Ω. In the following we restrict ϱ to $\Omega \times X$. Then

$$\rho_s(x) = \int_\Omega \rho(t,x)\, d\mu$$

is a modular in X_{ρ_s} (see [4]).. The respective modular space will be denoted

$$X_{\rho_s} = \{x: \quad \rho_s(\lambda x) \longrightarrow 0 \quad \text{as } \lambda \to 0,\ x \in X\};$$

then

$$|x|_{\rho_s} = \inf \left\{ u > 0: \rho\left(\frac{x}{u}\right) \leqslant u \right\}$$

is an F-norm in X_{ρ_s} (see [3]). In case ρ is convex, i.e. if $\rho(t, \alpha x + \beta y) \leqslant \alpha\rho(t,x) + \beta\rho(t,y)$ for $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$, for all $t \in \Omega$, where $A \in \Sigma$ is a fixed set of measure 0,

$$\| x \|_{\rho_s} = \inf \left\{ u > 0: \rho\left(\frac{x}{u}\right) \leqslant 1 \right\}$$

is a norm in X_{ρ_s}, equivalent to the former one. It is easily seen that an element $x \in X$ belongs to X_{ρ_s}, if and only if, there exists a $\lambda_0 > 0$ such that $\rho(\lambda_0 x) < \infty$ (see [1]).

1.1. The following example shows the connection between the above notions and integral transforms. We take a function $k: \Omega \times \Omega \times \langle 0,\infty) \to \langle 0, \infty)$, called kernel, supposing $k(t,s,u)$ to be measurable in $\Omega \times \Omega \times \langle 0, \infty)$, continuous and strictly increasing in u for all $(t,s) \in \Omega \times \Omega$, $k(t,s,0) = 0$. Then

$$\rho(t,x) = \int_\Omega k(t,s,|x(s)|)\, d\mu(s)$$

satisfies the above conditions. If we suppose that $\int_\Omega k(t,s,u)\, d\mu(t) > 0$ for all $u > 0$ and a.e. $s \in \Omega$, then the space X_{ρ_s} is complete ([1], Th. 6). This was applied in [1] and [2], in order to solve the equation $x(t) = \varkappa\, \rho(t,x)$ in X_{ρ_s}.

1.2. In this paper, the following properties of ρ will be needed. ρ will be called local in X, if $A \in \Sigma$

and $\mu(A) < \infty$ imply $\chi_A \in X$, where χ_A is the characteristic function of the set A. ϱ will be called absolutely continuous at $x \in \mathfrak{X}$, if there exists a set $\Omega_1 \in \Sigma$ with $\mu(\Omega_1) = 0$ such that for every $\varepsilon > 0$ there is a $\delta > 0$ such that for any $t \in \Omega \setminus \Omega_1$ and every $B \in \Sigma$, the inequality $\mu(B) < \delta$ implies $\varrho(t, x\chi_B) < \varepsilon$. ϱ will be called approximately finite at $x \in \mathfrak{X}$, if there exists a set $\Omega_1 \in \Sigma$ with $\mu(\Omega_1) = 0$ such that for every $\varepsilon > 0$ there is a set $A \in \Sigma$, $\mu(A) < \infty$, such that for any $t \in \Omega \setminus \Omega_1$ there holds $\varrho(t, x\chi_{\Omega \setminus A}) < \varepsilon$. Finally, we shall say that ϱ is regular, if ϱ is local in X, absolutely continuous and approximately finite at each element of X_{ϱ_s}, and if $x \in X_{\varrho_s}$ implies $|x(t)| < \infty$ a.e. in Ω.

2. We are going now to investigate the subspace E_{ϱ_s} of finite elements of X_{ϱ_s}. An element $x \in X_{\varrho_s}$ will be called finite, if $\varrho_s(\lambda x) < \infty$ for every $\lambda > 0$. E_{ϱ_s} will denote the space of all finite elements of X_{ϱ_s}. Obviously, E_{ϱ_s} is a linear, closed subspace of X_{ϱ_s}. Moreover, if $y \in E_{\varrho_s}$, x is Σ - measurable and $|x(t)| \leqslant |y(t)|$ a.e. in Ω, then $x \in E_{\varrho_s}$.

2.1. Lemma. Let ϱ be regular, $y_n \in \mathfrak{X}$ for $n = 1, 2, \ldots$, $z \in E_{\varrho_s}$. Moreover, let $0 \leqslant y_n(t) \to 0$ as $n \to \infty$ and $y_n(t) \leqslant z(t)$ a.e. in Ω. Then $\varrho(t, y_n) \to 0$ as $n \to \infty$ a.e. in Ω.

Proof. Let us take an arbitrary $\varepsilon > 0$. Let $\Omega_1 \in \Sigma$, $\delta > 0$ and $A \in \Sigma$ have the same meaning as in 1.2. By Egoroff's theorem, there is a set $B \in \Sigma$ with $B \subset A$, $\mu(B) < \delta$ for which $y_n(t) \to 0$ uniformly in $A \setminus B$. Since $0 \leqslant y_n(t) \leqslant z(t)$ a.e., applying the properties of a pseudomodular we obtain easily $\varrho(t, y_n) \leqslant 2\varepsilon + \varrho(t, 3y_n \chi_{A \setminus B})$ a.e. in Ω. Let $\eta > 0$ be arbitrary, then we may choose an index N such that $3y_n(t)\chi_{A \setminus B}(t) \leqslant \eta \chi_A(t)$ for $n > N$ and $t \in A \setminus B$. Hence $\varrho(t, y_n) \leqslant 2\varepsilon + \varrho(t, \eta \chi_A)$ for $n > N$ a.e. in Ω, say for $t \in \Omega \setminus \Omega_2$ with $\mu(\Omega_2) = 0$. Let us fix $t \in \Omega \setminus \Omega_2$, then $\varrho(t, \eta\chi_A) < \varepsilon$ for sufficiently small

$\eta > 0$. Hence $\varrho(t, y_n) < 3\varepsilon$ for sufficiently large n.

2.2. Theorem. If ϱ is regular, then the set S_{ϱ_s} of simple functions in E_{ϱ_s} is dense in E_{ϱ_s}.

Proof. Let us first suppose that $x \in E_{\varrho_s}$, $x(t) \geqslant 0$ a.e. in Ω. Let (x_n) be a nondecreasing sequence of non-negative simple functions such that $x_n(t) \to x(t)$ for $t \in \Omega$. Then $x_n \in S_{\varrho_s}$. Let us take any $\lambda > 0$. Then the sequence of functions $y_n = \lambda(x - x_n)$ satisfies the assumptions of Lemma 2.1 with $z = \lambda(x - x_1)$. Hence $\varrho(t, \lambda(x - x_n)) \to 0$ as $n \to \infty$ a.e. in Ω. Moreover, $\varrho(t, \lambda(x - x_n)) \leqslant \varrho(t, \lambda(x - x_1))$ and $\varrho(t, \lambda(x - x_1))$ is integrable over Ω. Hence $\dot{\varrho}_s(\lambda(x - x_n)) \to 0$ as $n \to \infty$. Since $\lambda > 0$ is arbitrary, we conclude that $x_n \to x$ in X_{ϱ_s}. Now, if $x \in E_{\varrho_s}$ is arbitrary, we write x in the form $x = x_+ - x_-$, where x_+ and x_- are the positive part and the negative part of x, respectively. Since $x_+, x_- \in E_{\varrho_s}$, we may apply the former part of the proof.

2.3. We shall say that the measure μ is finitely separable, if there exists a sequence of sets $A_n \in \Sigma$ with $\mu(A_n) < \infty$ possessing the following property: for every $A \in \Sigma$ such that $\mu(A) < \infty$ there exists a nondecreasing suquence of indices (n_i) for which $\mu(A_{n_i} \div A) \to 0$ as $i \to \infty$.

Let us remark that taking (A_{n_i}) in such a menner that $\mu(A_{n_i} \div A) < 2^{-i}$ for $i = 1, 2, \ldots$ and $B = A \cup \bigcup_{i=1}^{\infty} (A - A_{n_i})$, we obtain $B \in \Sigma$, $\mu(B) < \infty$, $A \subset B$, $A_{n_i} \subset B$ for $i = 1, 2, \ldots$

2.4. Theorem. Let ϱ be regular and let for any given $A \in \Sigma$ the condition $\chi_A \in E_{\varrho_s}$ be equivalent to $\mu(A) < \infty$. If the measure μ is finitely separable, then E_{ϱ_s} is a separable subspace of X_{ϱ_s}.

Proof. By 2.2, it is sufficient to show that the characteristic function χ_A of any set $A \in \Sigma$ with $\mu(A) < \infty$

may be approximated in X_{ϱ_s} as well as we please by characteristic functions of sets A_n from 2.3. Given $A \in \Sigma$, $\mu(A) < \infty$, let us take the set B and the sequence (n_i) as in 2.3. Let $0 < \eta < 1$, then

$$\mu(\{t: \chi_{A_{n_i} \dot{-} A}(t) \geqslant \eta,\ t \in \Omega\}) = \mu(A_{n_i} \dot{-} A) \to 0 \text{ as } i \to \infty.$$

Hence $\chi_{A_{n_i} \dot{-} A}(t) \to 0$ in measure μ. Let us take an arbitrary $\lambda > 0$ and let $y_i = \lambda \chi_{A_{n_i} \dot{-} A}$. One may find an increasing sequence of indices (i_r) independent of λ such that $y_{i_r}(t) \to 0$ a.e. in Ω. Moreover, $0 \leqslant y_{i_r}(t) \leqslant \lambda \chi_B(t)$ for $t \in \Omega$. Applying 2.1 we get $\varrho(t, y_{i_r}) \to 0$ a.e. in Ω. Moreover, $\varrho(t, y_{i_r}) \leqslant \varrho(t, \lambda\chi_B)$ and $\int_\Omega \varrho(t, \lambda\chi_B)\, d\mu =$ $= \varrho_s(\lambda\chi_B) < \infty$. Hence

$$\varrho_s(\lambda(\chi_{A_{n_{i_r}}} - \chi_A)) = \int_\Omega \varrho(t, y_{i_r})\, d\mu \to 0 \quad \text{as } r \to \infty.$$

Consequently, $\chi_{A_{n_{i_r}}} \to \chi_A$ in X_{ϱ_s}.

3. In this Section we shall suppose that $\Omega \subset \Omega_0$, where Ω_0 is a group with an operation +. Moreover, we shall suppose that if $A \in \Sigma$, then $(A + t) \cap \Omega \in \Sigma$ for any $t \in \Omega_0$. For an arbitrary function x on Ω we shall write $x_0(t) =$ $= x(t)$ for $t \in \Omega$, $x_0(t) = 0$ for $t \in \Omega_0 \setminus \Omega$. Moreover, if y is a function defined on Ω_0, the restriction of y to Ω will be denoted by $y|\Omega$. We shall write $x_0(\cdot) = x_0$ and $x_0(\cdot + h) = y$, where $y(t) = x_0(t + h)$, $h \in \Omega_0$.

3.1. We shall say that ϱ is translation semiinvariant at $x \in X$, if there is a constant $K > 0$ such that for all $t, h \in \Omega_0$ and for every $\lambda > 0$ there holds the inequality

$$(*) \qquad \varrho(t, \lambda x_0(\cdot + h)|\Omega) \leqslant \varrho(t, K\lambda x).$$

3.2. It is easily observed that if ϱ is convex, then ϱ is translation semiinvariant at x, if and only if, there exist constants $K_1, K_2 > 0$ such that for all $t, h \in \Omega_0$ and for every $\lambda > 0$ there holds the inequality

$$\varrho(t, \lambda x_0(\cdot + h)|\Omega) \leq K_1 \varrho(t, K_2 \lambda x),$$

because $K_1 \varrho(t, K_2 \lambda x) \leqslant \varrho(t, K_1, K_2 \lambda x)$ for $K_1 > 1$.

Also, if ϱ is convex, $x(t)$, $y(t) \geqslant 0$ a.e. in Ω and ϱ is translation semiinvariant both at x and y , then ϱ is translation semiinvariant at x + y, too.

3.3. Lemma. Let ϱ be translation semiinvariant at x. If $x \in X_{\varrho_s}$, then $x_0(\cdot + h)|\Omega \in X_{\varrho_s}$ and $[x_0(\cdot + h) - x_0(\cdot)]|\Omega \in X_{\varrho_s}$. If $x \in E_{\varrho_s}$, then $x_0(\cdot + h)|\Omega \in E_{\varrho_s}$ and $[x_0(\cdot + h) - x_0(\cdot)]|\Omega \in E_{\varrho_s}$.

Proof. Let $x \in X_{\varrho_s}$ and let $\lambda_0 > 0$ be chosen in such a manner that $\varrho_s(\lambda_0 x) < \infty$. Integrating the inequality $(*)$ over Ω we get for $0 \leqslant \lambda \leqslant \lambda_0/K$

$$\varrho_s(\lambda x_0(\cdot + h)|\Omega) \leqslant \varrho_s(K\lambda x) \leqslant \varrho_s(\lambda_0 x) < \infty,$$

and so $x_0(\cdot + h)|\Omega \in X_{\varrho_s}$. Other parts of the lemma are shown similarly.

3.4. We shall investigate now the translation operator T_h defined for $x \in X_{\varrho_s}$ and $h \in \Omega_0$ as follows:

$$(T_h x)(t) = x_0(t + h)|\Omega \quad \text{for } t \in \Omega.$$

3.5. Theorem. Let ϱ be convex and translation semiinvariant at $x \in X_{\varrho_s}$ with a constant $K > 0$. Then $T_h \in X_{\varrho_s}$ and $\|T_h x\|_{\varrho_s} \leq K \|x\|_{\varrho_s}$ for all $h \in \Omega_0$.

Proof. By Lemma 3.3, we have $T_h x \in X_{\varrho_s}$. Integrating the inequality $(*)$ over Ω with $\lambda = 1/\eta$, we get

$$\varrho_s\left(\frac{T_h x}{\eta}\right) = \varrho_s\left(\frac{x_0(\cdot + h)|\Omega}{\eta}\right) \leqslant \varrho_s\left(\frac{Kx}{\eta}\right) \text{ for any } \eta > 0.$$

Consequently, we obtain $\|T_h x\|_{\varrho_s} \leqslant K \|x\|_{\varrho_s}$.

3.6. As an example, let us take ρ defined by means of a kernel k as in 1.1, where $\Omega \subset \mathbb{R}^p$ and μ is the Lebesgue measure. Let $x \in X$ and let us suppose that there exists a constant $K > 0$ such that

$$\int_{(\Omega - h)\cap\Omega} k(t, s, \lambda \mid x(s + h)| \,)\, ds \leqslant \int_{\Omega} k(t, s, K\lambda |x(s)|)\, ds$$

for all $t, h \in \mathbb{R}^p$ and every $\lambda > 0$. Then ρ is translation semiinvariant at x.

4. In this Section, Ω will mean a Lebesgue measurable subset of $\Omega_0 = \mathbb{R}^p$, Σ - the σ - algebra of all Lebesgue measurable subsets of Ω, and μ - the Lebesgue measure; we shall then write m in place of μ. Let $\mathcal{P}$ be the family of all sets of the form

$$P = \langle \alpha_1, \beta_1 \rangle \times \ldots \times \langle \alpha_p, \beta_p \rangle \subset \Omega,$$

where $\alpha_i < \beta_i$ for $i = 1, 2, \ldots, p$.

4.1. Lemma. Let $P \in \mathcal{P}$ and let $x = \chi_P$ be the characteristic function of the set P in Ω. Let

$$y(t) = x_0(t + h) - x_0(t + k),$$

where $h, k \in \mathbb{R}^p$, $h = (h_1, \ldots, h_p)$, $k = (k_1, \ldots, k_p)$, $|h_i - k_i| < \beta_i - \alpha_i$ for $i = 1, 2, \ldots, p$. Finally, let

$$B = \{t: \ y(t) \neq 0, \ t \in \mathbb{R}^p\}.$$

Then

$$m(B) \leqslant \frac{2^{p+1} V}{a} \, |h - k|,$$

where $V = (\beta_1 - \alpha_1) \cdot \ldots \cdot (\beta_p - \alpha_p)$, $a = \min_i (\beta_i - \alpha_i)$ and $|h - k|$ is the euclidean distance between h and k in $\mathbb{R}^p$.

Proof. It is easily seen that $y(t) = 1$ iff $t \in P - h$ and $t \in P - k$, $y(t) = -1$ iff $t \in P - k$ and $t \in P - h$, and $y(t) = 0$ elsewhere. This shows that denoting $P_h = P - h$, $P_k = P - k$, we have $B = (P_h \cap P_k') \cup (P_h' \cap P_k)$, where the prime denotes the complement of the set with respect to Ω, and $m(B) = m(P_h \cap P_k') + m(P_k \cap P_h')$. Moreover, it is easily observed that both sets $P_h \cap P_k'$ and $P_k \cap P_h'$ are contained in the set $Q_1 \setminus Q_2$, where

$$Q_1 = \langle \alpha_1 - |h_1-k_1|, \beta_1+|h_1-k_1|\rangle \times \ldots \times \langle \alpha_p - |h_p-k_p|, \beta_p+|h_p-k_p|\rangle,$$

$$Q_2 = \langle \alpha_1 + |h_1-k_1|, \beta_1-|h_1-k_1|\rangle \times \ldots \times \langle \alpha_p + |h_p-k_p|, \beta_p-|h_p-k_p|\rangle.$$

Writing $a_i = \beta_i - \alpha_i$ and $d_i = |h_i - k_i|$ for $i = 1,2,\ldots,p$, $r = \max\limits_i d_i/a_i$, we have then

$$m(B) \leq 2\, m(Q_1 \setminus Q_2) \leq 2V\left[\left(1+\frac{d_1}{a_1}\right)\cdot\ldots\cdot\left(1+\frac{d_p}{a_p}\right) - \left(1-\frac{d_1}{a_1}\right)\cdot\right.$$

$$\left.\cdot\ldots\cdot\left(1-\frac{d_p}{a_p}\right)\right] \leq 2V\left[(1+r)^p - (1-r)^p\right] \leq 2^{p+1}Vr \leq \frac{2^{p+1}V}{a}\,|h-k|.$$

4.2. Theorem. Let $m(\Omega) < \infty$ or $\chi_\Omega \in E_{\varrho_s}$ and let ϱ be absolutely continuous at any constant function. Let $f(h) = T_h \chi_P$ for a fixed $P \in \mathcal{P}$. Then the map $f: \mathbb{R}^p \longrightarrow X_{\varrho_s}$ is uniformly continuous in the norm $\| \|_{\varrho_s}$, and in case of ϱ convex also in the norm $\| \|_{\varrho_s}$.

Proof. First, we suppose $m(\Omega) < \infty$. Let $\varepsilon > 0$ and $\lambda > 0$ be given and let us choose Ω_1 and $\delta > 0$ according to the definition of absolute continuity at $x = \lambda\chi_\Omega$, with $\varepsilon/m(\Omega)$ in place of ε. Let us suppose that $|h - k| < \eta$, where $\eta = \frac{a}{2^{p+1}V}$, a and V being defined in 4.1. Then the set B from 4.1 satisfies the inequality $m(B) < \delta$. Consequently, $\varrho(t, \lambda\chi_B|\Omega) < \varepsilon/m(\Omega)$ for $t \in \Omega \setminus \Omega_1$. Hence

$$\varrho_s[\lambda(T_h\chi_P - T_k\chi_P)] = \int_\Omega \varrho(t, \lambda\chi_B|\Omega)\, dt < \varepsilon$$

for $|h - k| < \eta$. New, let us suppose $\varepsilon < 1$ and $\lambda = 1/\varepsilon$, then

$$\varrho_s\left(\frac{T_h\chi_P - T_k\lambda_P}{\varepsilon}\right) < \varepsilon < 1 \quad \text{for } |h - k| < \eta.$$

Thus, $| T_h\chi_P - T_k\chi_P|_{\varrho_s} < \varepsilon$ for $|h - k| < \eta$, and in case of convex ϱ, also $\| T_h\chi_P - T_k\chi_P\|_{\varrho_s} < \varepsilon$ for $|h - k| < \eta$.

New, let us suppose that $\chi_\Omega \in E_{\varrho_s}$, then $\varrho(t, 2\lambda\chi_\Omega)$ is integrable in Ω for every $\lambda > 0$. Hence there exists a set $A \in \Sigma$, $m(A) < \infty$, such that $\int\limits_{\Omega \setminus A} \varrho(t, 2\lambda\chi_\Omega)dt < \frac{\varepsilon}{2}$.

Now, applying the first part of the proof with A in place of Ω and 2λ and $\frac{\varepsilon}{2}$ in place of λ and ε, respectively, we obtain

$$\varrho_s[\lambda(T_h\chi_P - T_k\chi_P)] \leqslant \int_\Omega \varrho(t, \lambda\chi_B)\,dt \leqslant$$
$$\leqslant \int_A \varrho(t, 2\lambda\chi_B)\,dt + \int_{\Omega\setminus A} \varrho(t, 2\lambda\chi_\Omega)\,dt < \varepsilon$$

for $|h - k| < \eta$, and the result follows as in the first part of the proof.

4.3. Corollary. Let $m(\Omega) < \infty$ or $\chi_\Omega \in E_{\varrho_s}$ and let ϱ be absolutely continuous at any constant function. Finally, let $x = \sum_{j=1}^{r} d_j\chi_{P_j}$, where $P_j \in \mathcal{P}$, and $f(h) = T_h x$. Then the map $f: \mathbb{R}^p \longrightarrow X_{\varrho_s}$ is uniformly continuous in the norm $|\ |_{\varrho_s}$, and in case of ϱ convex also in the norm $\|\ \|_{\varrho_s}$.

4.4. Theorem. Let $\Omega \in \mathbb{R}^p$ be an open set, $m(\Omega) < \infty$ or $\chi_\Omega \in E_{\varrho_s}$, and let ϱ be absolutely continuous at any constant function. If $E \in \Sigma$, $m(E) < \infty$, then for every $\varepsilon > 0$ there exist sets $P_1, P_2, \dots, P_n \in \mathcal{P}$ with pairwise disjoint interiors such that $|\chi_E - \chi_{\bigcup_{i=1}^{n} P_i}|_{\varrho_s} < \varepsilon$. If ϱ is convex, $|\ |_{\varrho_s}$ may be replaced by $\|\ \|_{\varrho_s}$.

Proof. First, let us suppose that $m(\Omega) < \infty$. Given $\varepsilon > 0$ and $\lambda > 0$, we apply the definition of absolute continuity at $x = \lambda\chi_\Omega$ with $\varepsilon/m(\Omega)$ in place of ε. Let Ω_1 and $\delta > 0$ be chosen, accordingly. There exists an open set $G \subset \Omega$ such that $E \subset G$ and $m(G \setminus E) < \frac{\delta}{2}$. Then G can be written in the form $G = \bigcup_{i=1}^{\infty} P_i$, with $P_i \in \mathcal{P}$, where $P_1, P_2, \dots$ have pairwise disjoint interiors, and $\sum_{i=1}^{\infty} m(P_i) = m(G) < \infty$

Hence $\sum_{i=n+1}^{\infty} m(P_i) < \frac{\delta}{2}$ for an index n. Let us write $E_\varepsilon = \bigcup_{i=1}^{n} P_i$, then $m(G \setminus E_\varepsilon) < \frac{\delta}{2}$. Thus, the symmetric difference $E \dot{-} E_\varepsilon$ satisfies the inequality

$m(E \doteq E_\varepsilon) \leq m(G \setminus E_\varepsilon) + m(G \setminus E) < \delta$. Hence

$$\varrho_s[\lambda(\chi_E - \chi_{E_\varepsilon})] = \int_\Omega \varrho(t, \lambda\chi_{E \doteq E_\varepsilon})\, dt < \varepsilon .$$

Choosing $\varepsilon < 1$ and $\lambda = 1/\varepsilon$, we conclude $|\chi_E - \chi_{E_\varepsilon}|_{\varrho_s} < \varepsilon$ or $\|\chi_E - \chi_{E_\varepsilon}\|_{\varrho_s} < \varepsilon$ as in the proof of 4.2.

If we suppose $\chi_\Omega \in E_{\varrho_s}$, and we argue as in the proof of 4.2, we obtain

$$\varrho_s[\lambda(\chi_E - \chi_{E_\varepsilon})] \leq \int_A \varrho(t, 2\lambda\ \chi_{E \doteq E_\varepsilon})\, dt + \int_{\Omega \setminus A} \varrho(t, 2\lambda\chi_\Omega)\, dt < \varepsilon$$

for suitably chosen $P_1, P_2, \ldots, P_n \in \mathcal{P}$.

4.5. Corollary. Let $\Omega \subset \mathbb{R}^r$ be open. Let $m(\Omega) < \infty$ or $\chi_\Omega \in E_{\varrho_s}$. Moreover, let ϱ be absolutely continuous at any constant function. If $x = \sum_{i=1}^{q} c_i \chi_{E_i}$, where $E_i \in \Sigma$, $m(E_i) < \infty$, then for every $\varepsilon > 0$ there exist numbers $d_1, d_2, \ldots, d_r$ and sets $P_1, P_2, \ldots, P_r \in \mathcal{P}$ with pairwise disjoint interiors such that $|x - \sum_{j=1}^{r} d_j \chi_{P_j}|_{\varrho_s} < \varepsilon$. If ϱ is convex, $|\ |_{\varrho_s}$ may be replaced by $\|\ \|_{\varrho_s}$.

Applying both corollaries 4.3, 4.5 and theorems 3.5 and 2.2, we shall prove the following theorem on uniform continuity of $f(h) = T_h x$:

4.6. Theorem. Let $\Omega \subset \mathbb{R}^r$ be open, $m(\Omega) < \infty$ and let ϱ be convex, regular, alsolutely continuous at any constant function and translation semiinvariant at every $x \in X_{\varrho_s}$ with a constant $K > 0$ independent of x. Then the map $f: \mathbb{R}^r \to X_{\varrho_s}$ defined by $f(h) = T_h x$ is uniformly continuous in the norm $\|\ \|_{\varrho_s}$, provided $x \in E_{\varrho_s}$.

Proof. By 3.5, we have $\|T_h u\|_{\varrho_s} \leq K \|u\|_{\varrho_s}$ for every $u \in X_{\varrho_s}$. Let $x \in E_{\varrho_s}$ be given. By 2.2 and 4.5, there exists a function $z = \sum_{j=1}^{r} d_j \chi_{P_j}$ with $P_j \in \mathcal{P}$ such that $\|x - z\|_{\varrho_s} < \varepsilon/3K$.

Hence

$$\|f(h) - f(k)\|_{\varrho_s} \leqslant 2K \| x - z \|_{\varrho_s} + \| T_h z - T_k z \|_{\varrho_s} < \frac{2}{3}\varepsilon +$$

$+ \| T_h z - T_k z \|_{\varrho_s}$. By 4.3, $T_h z : \mathbb{R}^p \to X_{\varrho_s}$ is uniformly continuous. Hence there exists $\eta > 0$ such that if $|h - k| < \eta$, then $\| T_h z - T_k z \|_{\varrho_s} < \frac{\varepsilon}{3}$. Consequently, if $|h - k| < \eta$, then $\| f(h) - f(k) \|_{\varrho_s} < \varepsilon$.

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SOME THEOREMS ON GENERALIZED MODULAR SPACES

Abstract

Let X_{ϱ_s} be the modular space generated by the modular $\varrho_s(x) = \int_\Omega \varrho(t, x)\, d\mu$, x - any measurable function over Ω , and let E_{ϱ_s} be the subspace of finite elements of X_{ϱ_s}. There are considered problems of density of simple functions in E_{ϱ_s} and of separability of E_{ϱ_s} . This is applied in case $\Omega \subset \mathbb{R}^p$ in order to investigate the problem of uniform continuity of the translation operator $T_h x$ with respect to $h \in \mathbb{R}^p$.

O PEWNYCH TWIERDZENIACH O UOGÓLNIONYCH PRZESTRZENIACH MODULARNYCH

Niech X_{ϱ_s} będzie przestrzenią modularną, generowaną poprzez modular $\varrho_s(x) = \int_{\Omega} \varrho(t,x)\, d\mu$, gdzie x jest funkcją mierzalną nad Ω i niech E_{ϱ_s} będzie podprzestrzenią elementów skończonych z X_{ϱ_s}. Rozważa się problem gęstości funkcji prostych w E_{ϱ_s} oraz ośrodkowość E_{ϱ_s}. Rozważa się to w przypadku, gdy $\Omega \subset R^p$, aby zbadać problem jednostajnej ciągłości operatora przesunięcia $T_h x$ ze względu na $h \in R^p$.