ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ W BYDGOSZCZY
Problemy Matematyczne 1980 z. 2

Jerzy Norwa
Warszawa

ON A CERTAIN ALGORITHM OF COMPUTING $\sqrt[k]{D}$

1. It is known (cf. [1] § 99) that a sequence (x_n) defined recursively by

(1) $$x_{n+1} = \frac{1}{2}\left(x_n + \frac{D}{x_n}\right), \text{ where } x_0 > 0 \text{ and } D > 0,$$

has the limit

(2) $$\lim_{n\to\infty}\left(\frac{1}{2}\left(x_n + \frac{D}{x_n}\right)\right) = \sqrt{D} .$$

The generalisation of the formula (2) to the case of $\sqrt[k]{D}$ is not considered in [1], but the following Hobson's formula

(3) $$\lim_{n\to\infty}\left(x_n + \frac{D-x_n^{k-1}}{k(x_n+1)^{k-1}}\right) = \sqrt[k]{D} \quad , x_0^k \leq D < (x_0+1)^k, \quad k\in N_2^{x)}$$

is proved[xx].

It is proved [3] that the sequence (x_n) defined recursively by

(4) $$x_{n+1} = \frac{1}{3}\left(2x_n + \frac{D}{x_n^2}\right), \text{ where } x_0 > 0 \text{ and } D > 0,$$

has the limit

---

x) We adopt the notation $N_0 = \{0,1,2,\ldots\}$ and $N_k = \{k,k+1,k+2,\ldots\}$ for $k\in N_0$.

xx) Vide [2], § 36.

(5) $$\lim_{n\to\infty}\left(\frac{1}{3}\left(2x_n+\frac{D}{x_n^2}\right)\right)=\sqrt[3]{D}\ .$$

2.1. Let $x_0>0$, $D>0$, $k\in N_2$, and let (x_n) be the sequence defined recursively by

(6) $$x_{n+1}=\frac{1}{k}\left[(k-1)x_n+\frac{D}{x_n^{k-1}}\right]\ .$$

The purpose of this note is to prove that the sequence (6) converges to $\sqrt[k]{D}$.

If $x_0=\sqrt[k]{D}$ then it follows from (6) that $x_n=\sqrt[k]{D}$ for all $n\in N_0$. In the sequel we assume that $x_0\neq\sqrt[k]{D}$. From $x_0>0$ it follows that $x_n>0$ for every $k\in N_2$ and every $n\in N_0$.

2.2. We shall prove that the sequence (6) is bounded from below. We have

(7) $$x_{n+1}-\sqrt[k]{D}=\frac{1}{kx_n^{k-1}}\left[(k-1)x_n^k-kx_n^{k-1}\sqrt[k]{D}+D\right].$$

Let us note that $x=\sqrt[k]{D}$ is a double root of the polynomial

(8) $$f(x)=(k-1)x^k-kx^{k-1}\sqrt[k]{D}+D.$$

Hence we may assume that

(9) $$f(x)=(x-\sqrt[k]{D})^2\sum_{j=0}^{k-2}A_j^k x^{k-j-2}\ .$$

By the method of undetermined coefficients we find

(10) $$A_0^k=k-1,\quad A_1^k=(k-2)\sqrt[k]{D}\ ,\dots,A_j^k=2A_{j-1}^k\sqrt[k]{D}-A_{j-2}^k\sqrt[k]{D^2},\dots A_{k-2}^k=\sqrt[k]{D^{k-2}},$$

where $j\in\{2,3,4,\dots(k-3)\}$.

It is easy to prove by induction that

(11) $$A_j^k=(k-j-1)\sqrt[k]{D^j}\ ,\ \text{for}\ \ j\in\{0,1,2,\dots(k-2)\}.$$

Hence

(12) $$(k-1)x^k-kx^{k-1}\sqrt[k]{D}+D\equiv(x-\sqrt[k]{D})^2\sum_{j=0}^{k-2}(k-j-1)\sqrt[k]{D^j}\ x^{k-j-2}\ .$$

In view of (12) the formula (7) can be written in the form

$$x_{n+1}-\sqrt[k]{D} = \frac{(x_n-\sqrt[k]{D})^2}{kx_n^{k-1}} \sum_{j=0}^{k-2}(k-j-1)\sqrt[k]{D^j}\, x_n^{k-j-2} \; . \tag{13}$$

Since $x_n > 0$ for every $n \in N_0$ and $A_j^k > 0$ for every $j \in \{0,1,2,\ldots (k-2)\}$ it follows from (13) that

$$x_{n+1} > \sqrt[k]{D} \tag{14}$$

for every $n \in N_0$.

Consequently, the sequence (6) is bounded from below.

2.3. The sequence $(x_1, x_2, x_3, \ldots)$ is decreasing because

$$x_{n+1}-x_n = \frac{D-x_n^k}{kx_n^{k-1}}$$

and by (14)

$$x_{n+1} < x_n \tag{15}$$

for every $n \in N_1$.

It follows from (14) and (15) that the sequence (6) is convergent, i.e. there exists the limit $\lim\limits_{n\to\infty}(x_n) = q$ satisfying the conditions

$$q = \frac{1}{k}\left[(k-1)q + \frac{D}{q^{k-1}}\right], \quad q > 0.$$

It follows that $q = \sqrt[k]{D}$, i.e.

$$\lim_{n\to\infty}\left(\frac{1}{k}\left[(k-1)x_n + \frac{D}{x_n^{k-1}}\right]\right) = \sqrt[k]{D} \; . \tag{16}$$

3. We shall estimate the error of the n-th approximation $\beta_n = x_n - \sqrt[k]{D}$.

It follows from (14) and from the identity $\sqrt[k]{D} = \dfrac{D}{\sqrt[k]{D^{k-1}}}$

that the condition $x_n > \sqrt[k]{D}$ implies $\sqrt[k]{D} > \frac{D}{x_n^{k-1}}$ and

$$x_n - \sqrt[k]{D} < x_n - \frac{D}{x_n^{k-1}}.$$

Consequently

(17) $$0 \leqslant \beta_n < x_n - \frac{D}{x_n^{k-1}}.$$

Suppose

(18) $$D \geqslant \frac{1}{2^k}.$$

In this case β_{n+1} can be estimated in terms of β_n. In fact, by (13) and (18) we have

$$\beta_{n+1} = \frac{\beta_n^2}{kx_n^{k-1}} \sum_{j=0}^{k-2} (k-j-1)\sqrt[k]{D^j}\ x_n^{k-j-2}.$$

Substituting x_n in place of $\sqrt[k]{D}$ and using (14) we obtain

$$\beta_{n+1} < \frac{\beta_n^2}{kx_n^{k-1}} \sum_{j=0}^{k-2} (k-j-1)x_n^{k-2}$$

i.e.

$$\beta_{n+1} < \frac{\beta_n^2}{kx_n} \sum_{j=0}^{k-2} (k-j-1).$$

Since $\sum_{j=0}^{k-2} (k-j-1) = \frac{1}{2}k(k-1)$ and in virtue of (18) and (14) $x_n > \frac{1}{2}$, we get

(19) $$\beta_{n+1} < (k-1)\beta_n^2.$$

In particular, if $\beta_n = 10^{-s}$, where $s \in N_1$, then

(20) $$\beta_{n+1} < (k-1)10^{-2s}.$$

For $k=2$ we have $\beta_{n+1} < 10^{-2s}$, and for any integer $k \in [10^{p-1}+2,\ 10^p+1]$, where $p \in N_1$, we have $\beta_{n+1} < 10^{-2s+p}$.

4. Using the formula (16) we can casily compute $\sqrt[k]{D}$ witl

the help of a digital computer. For example, the following results have been obtained with the use of Odra 1204.

a) For $k=2$, $D=2$, $x_0=1$
$x_1=1{,}5$; $x_2=1{,}416\ 666\ 667$; $x_3=1{,}414\ 215\ 686$;
$x_4=x_5=\ldots=1{,}414\ 213\ 562$.

b) For $k=2$, $D=2$, $x_0=2$
$x_1=1{,}5$; $x_2=1{,}416\ 666\ 667$; $x_3=1{,}414\ 215\ 686$;
$x_4=x_5=\ldots=1{,}414\ 213\ 562$.

c) For $k=2$, $D=4$, $x_0=1$
$x_1=2{,}5$; $x_2=2{,}05$; $x_3=2{,}000\ 669\ 756$;
$x_4=2{,}000\ 000\ 093$; $x_5=x_6=\ldots=2{,}000\ 000\ 000$.

d) For $k=10$, $D=245$, $x_0=1$
$x_1=25{,}4;\ldots x_4=18{,}51\ 660\ 000;\ldots x_7=13{,}49\ 860\ 140;\ldots$
$x_{29}=1{,}733\ 525\ 144$; $x_{30}=1{,}733\ 471\ 119$;
$x_{31}=x_{32}=\ldots=1{,}733\ 471\ 111$.

REFERENCES

[1] W.Sierpiński, Działania nieskończone, cz.I,wyd.II, Warszawa 1948.

[2] E.W.Hobson, The theory of functions of real variable and the theory of Fourier's series, Vol.1,4-th ed, New York 1957.

[3] В. М. Брадис, Извлечение квадратного и кубичеокого корней из чисел, "Математика в школе" № 1, 1961.

ON A CERTAIN ALGORITHM OF COMPUTING $\sqrt[k]{D}$

Summary

Let $x_0>0$, $D>0$, $k\in N_1=\{1,2,3,\ldots\}$ and let (x_n) be the sequence defined recursively by $x_{n+1}=\frac{1}{k}\left[(k-1)x_n+\frac{D}{x_n^{k-1}}\right]$.

The purpose of this note is to prove that $x_{n+1} < x_n$ for every $n \in N_1$ and the sequence (x_n) couverges to $\sqrt[k]{D}$ ($\sqrt[1]{D} = D$).

Suppose $D > \frac{1}{2^k}$. In this cose β_{n+1} can be estimated in terms of $\beta_n = x_n - \sqrt[k]{D}$. We have $\beta_{n+1} < (k-1)\beta_n^2$. In particular if $\beta_n = 10^{-s}$, where $s \in N_1$, then $\beta_{n+1} < (k-1)\, 10^{-2s}$. For k=2 we have $\beta_{n+1} < 10^{-2s}$ and for any integer $k \in [1+10^{p-1}, 1+10^p]$, where $p \in N_1$, we have $\beta_{n+1} < 10^{p-2s}$.

O PEWNYM ALGORYTMIE OBLICZANIA $\sqrt[k]{D}$

Streszczenie

Załóżmy, że $x_0 > 0$, $D > 0$ i $k \in N_1 = \{1,2,3,\ldots\}$ i ciąg (x_n) jest określony rekurencyjnie

$$x_{n+1} = \frac{1}{k}\left[(k-1)x_n + \frac{D}{x_n^{k-1}}\right].$$

W pracy udowodniono, że ciąg (x_n) jest malejący dla każdego $n \in N_1$ i ma granicę $\sqrt[k]{D}$ ($\sqrt[1]{D} = D$).

Załóżmy, że $D > \frac{1}{2^k}$. W tym przypadku x_{n+1} może być oszacowany za pomocą $\beta_n = x_n - \sqrt[k]{D}$. Mamy $\beta_{n+1} < (k-1)\beta_n^2$. W szczególności, jeżeli $\beta_n = 10^{-s}$, gdzie $s \in N_1$, wówczas $\beta_{n+1} < (k-1)\, 10^{-2s}$. Dla k=2 mamy $\beta_{n+1} < 10^{-2s}$ i dla pewnego całkowitego $k \in [1+10^{p-1}, 1+10^p]$, gdzie $p \in N_1$ mamy, że $\beta_{n+1} < 10^{p-2s}$.