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ON ABSOLUTE CONVERGENCE OF FOURIER TRANSFORM FOR FUNCTIONS OF TWO VARIABLES

We consider real or complex functions of two real variables $x = (x_1, x_2) \in R^2$, where R^2 denotes the 2-dimensional space with inner product $x\,y = x_1y_1 + x_2y_2$. The Fourier transform for $f \in L^1$ is of the form

(1) $$\hat{f}(x_1,x_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y_1,y_2)\, e^{-i(x_1y_1+x_2y_2)}\, dy_1\, dy_2 .$$

Denoting $dy = dy_1dy_2$ we can rewrite the formula (1) as

(2) $$\hat{f}\ x = \frac{1}{2\pi}\int_{R^2} f(y)\ e^{-ixy}\ dy.$$

In this paper, both the forms will be used. The integral on the right-side of (2) for an integrable function f is absolutely convergent, hence $\hat{f}(x)$ is well defined. Subsequently, we consider functions from L^2; then the formula (2) is not correct in general.

Let $f \in L^2$, $r > 0$. The function

(3) $$f_r(x) = \begin{cases} f(x) & \text{if } |x| \leq r \\ 0 & \text{if } |x| > r \end{cases}$$

is for every $r > 0$ locally integrable, and hence the Fourier transform (2) exist for the function (3). Let as denote

(4) $$\hat{f}_r(x) = \frac{1}{2\pi}\int_{R^2} f_r(y)\, e^{-ixy}\, dy = \frac{1}{2\pi}\int_{|y|\leq r} f(y)\ e^{-ixy}\ dy.$$

It is known (see [4], page 45), that the sequence $\hat{f}_r$ converges in norm in L^2 to $f \in L^2$. Hence the expressions (1) and (2) should be understood as

(5) $\lim\limits_{r\to\infty} \int_{R^2} |\hat{f} - \hat{f}_r|^2 \, dx = 0.$

The increment of the function f and the 2-th integral modulus of continuity are defined as follows:

(6) $\Delta^{(1,2)}(f,x,h) = \Delta^{(1,2)}(f,x_1,x_2,h_1,h_2) =$

$= f(x_1+h_1, x_2+h_2) - f(x_1+h_1, x_2) - f(x_1, x_2+h_2) +$

$+ f(x_1, x_2)$.

(7) $\omega_2^{(1,2)}(\delta_1, \delta_2) = \sup\limits_{\substack{h_i \leq \delta_i \\ i=1,2}} \left(\int_{R^2} |\Delta^{(1,2)}(f,x,h)|^2 \, dx \right)^{1/2}$

We now calculate the Fourier transform (2) for the function (6) . It is known that for $g \in L^1$ there holds the formula: $\hat{g}(x+h) = e^{ihx} \hat{g}(x)$. In [5] it is shown that this formula holds also for $g \in L^2$. Applying it and using the linearity of the Fourier transform it is easily to show that

(8) $\hat{\Delta}^{(1,2)}(f,x,h) = (e^{ih_1x_1} - 1)(e^{ih_2x_2} - 1) \hat{f}(x_1,x_2)$.

Hence

$|\hat{\Delta}^{(1,2)}(f,x,h)| = 4|\sin(h_1x_1/2)| \, |\sin(h_2x_2/2)| \, |f(x_1,x_2)|$.

Putting 2h in place of h, we obtain

(9) $|\hat{\Delta}^{(1,2)}(f,x,2h)| = 4|\sin h_1x_1| \, |\sin h_2x_2| \, |\hat{f}(x_1,x_2)|$.

We shall prove the following

T h e o r e m 1. Let $f \in L^2$, $0 < \beta < 2$, $\gamma_1 > 0$, $\gamma_2 > 0$.

We denote: $1 - \frac{1}{2}\beta + \gamma_1 = a$, $1 - \frac{1}{2}\beta + \gamma_2 = b$, $\frac{5}{2}\beta - 1 - \gamma_1 = c$ and $\frac{5}{2}\beta - 1 - \gamma_2 = d$. If the integral modulus of continuity (7) satisfies the inequality

$$\sum_n \sum_k \left[\omega_2^{(1,2)} \left(2^{-n}, 2^{-k}\right)\right]^\beta \left(2^{na} + 2^{nc}\right)\left(2^{kb} + 2^{kd}\right) < \infty$$

then

$$(10) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |x_1|^{\gamma_1} \quad |x_2|^{\gamma_2} \quad |\hat{f}(x_1,x_2)|^\beta \quad dx_1 \, dx_2 < \infty .$$

P r o o f. Taking into account (9) and the Parseval formula $\| \hat{g} \|_{L^2} = \| g \|_{L^2}$ we obtain

$$(11) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sin^2 h_1 x_1 \sin^2 h_2 x_2 \mid \hat{f}(x_1,x_2)|^2 \, dx_1 \, dx_2 \leq$$

$$\leq \frac{1}{C}\left[\omega_2^{(1,2)}(2h_1, 2h_2)\right]^2 .$$

Let E_i, $i = 1,2,3,4$, $n,k, = 1,2 \ldots$ denote the following subsets of the plane R^2:

$$E_1 = \left\{(x_1,x_2) : \quad 2^{n-1} \leq x_1 \leq 2^n ; \; 2^{k-1} \leq x_2 \leq 2^k \right\}$$

$$E_2 = \left\{(x_1,x_2) : \quad 2^{-n} \leq x_1 \leq 2^{-n+1} ; \quad 2^{k-1} \leq x_2 \leq 2^k \right\}$$

(12)

$$E_3 = \left\{(x_1,x_2) : \quad 2^{n-1} \leq x_1 \leq 2^n \quad ; \quad 2^{-k} \leq x_2 \leq 2^{-k+1} \right\}$$

$$E_4 = \left\{(x_1,x_2) : \quad 2^{-n} \leq x_1 \leq 2^{-n+1} ; \; 2^{-k} \leq x_2 \leq 2^{-k+1} \right\}$$

The inequality (11) implies all the more

$$(13)\quad \int_{E_1} \sin^2 h_1x_1 \sin^2 h_2x_2 \,|\hat{f}(x_1,x_2)|^2 \,dx_1 \,dx_2 \leq$$

$$\leq \frac{1}{16}\left[\omega^{(1,2)}_{2}(2h_1,\ 2h_2)\right]^2 .$$

We now put $h_1 = \pi/2^{n+1}$, $h_2 = \pi/2^{k+1}$. For $(x_1,x_2) \in E_1$ we have $\pi/4 \leq h_1x_1 \leq \pi/2$, hence $\sin^2 h_ix_i \geq 1/2$, $i = 1,2$.

Denoting $A = \omega^{(1,2)}_{2}(2^{-n},\ 2^{-k})$ and applying (13) we obtain

$$(14)\int_{E_1} |\hat{f}(x)|^2 \,dx \leq C A^2,$$

where C is a positive constant. Suppose $p,q > 1$, $1/p + 1/q = 1$ and put $p = \beta/2$. From Hölder inequality and relationship (14) we have

$$\int_{E_1} |\hat{f}(x)|^{\beta} \,dx \leq \left(\int_{E_1} |\hat{f}(x)|^2 \,dx\right)^{\beta/2} \cdot 2^{(n+k)\, 1-\beta/2}$$

$$\leq C_1 A^{\beta}\ 2^{(n+k)(1-\beta/2)}.$$

For $(x_1,x_2) \in E_1$ there hold inequalities

$$|x_1|^{\gamma_1} \leq 2^{n\gamma_1}, \quad |x_2|^{\gamma_2} \leq 2^{k\gamma_2}, \text{ hence}$$

$$\int_{E_1} |x_1|^{\gamma_1}\ |x_2|^{\gamma_2}\ |\hat{f}(x_1,x_2)|^{\beta} dx_1\, dx_2 \leq C_1\ A^{\beta}\ 2^{na}\, 2^{kb}.$$

Suming the above obtained inequality with respect to n and k we have

$$(15)\int_1^{\infty}\int_1^{\infty} |x_1|^{\gamma_1}\ |x_2|^{\gamma_2} |\hat{f}(x_1,x_2)|^{\beta} \,dx_1\, dx_2 \leq C_1 \sum_n \sum_k A^{\beta}\, 2^{na} \cdot$$

$$\cdot\, 2^{kb}.$$

We now consider the subset E_2, formula (12). Let h_1 and h_2 have the very same meaning as above, hence

$\pi/2^{n+1} \leq x_1h_1 \leq \pi/2^n$. The inequality $(\sin x)/x > \frac{2}{\pi}$ implies $\sin h_1x_1 \geqslant 2^{-2n}$, and hence $\sin^2 h_1x_1 \geqslant 2^{-4n}$. However $\sin^2 h_2x_2 \geqslant 1/2$ is just the same as in case of E_1. Hence and from (11) it follows

$$\int_{E_2} |\hat{f}(x)|^2 \, dx \leq (1/8)\, A^2 \,.\, 2^{4n} .$$

From this inequality and from Hölder inequality, taking into account the limits of integration we obtain

$$(16)\ \int_{E_1} |x_1|^{\gamma_1} |x_2|^{\gamma_2} |\hat{f}(x_1,x_2)|^{\beta} \, dx_1 \, dx_2 \leq C_2 A^{\beta} 2^{nc} . 2^{kb} .$$

Suming in n and k, we get

$$(17)\ \int_0^1 \int_1^\infty |x_1|^{\gamma_1} |x_2|^{\gamma_2} |\hat{f}(x_1,x_2)|^{\beta} \, dx_1 \, dx_2 \leq C_2 \sum_n \sum_k A^{\beta} 2^{nc} .\, 2^{kb} .$$

Analogous calculations for E_3 and E_4 give the following result, immediately

$$(18)\ \int_1^\infty \int_0^1 |x_1|^{\gamma_1} |x_2|^{\gamma_2} |\hat{f}(x_1,x_2)|^{\beta} \, dx_1 \, dx_2 \leq C_3 \sum_n \sum_k A^{\beta} 2^{na} \,.\, 2^{kd}$$

and

$$(19)\ \int_0^1 \int_0^1 |x_1|^{\gamma_1} |x_2|^{\gamma_2} |\hat{f}(x_1,x_2)|^{\beta} \, dx_1 \, dx_2 \leq C_4 \sum_n \sum_k A^{\beta} 2^{nc} .2^{kd} .$$

Adding the relationships (15), (17), (18), (19) and putting $C_5 = \max(C_1, C_2, C_3, C_4)$ we obtain

$$(20)\ \int_0^\infty \int_0^\infty |x_1|^{\gamma_1} |x_2|^{\gamma_2} |\hat{f}(x_1,x_2)|^{\beta} \, dx_1 \, dx_2 \leq$$
$$\leq \sum_n \sum_k A^{\beta} (2^{na} + 2^{nc}) (2^{kb} + 2^{kd}) .$$

Note. The integral (19) does not need to be included in the sum (20) , because it is finite. However, the estimate of (19) does not increase the estimate of the integrals (15) , (17) and (18) , therefore for the reason of symmetry on the rightside of (20) , the integral (19) was included in the sum (20) .

The right-side of (20) is finite from the assumption, hence

$$(21) \int_0^\infty\int_0^\infty |x_1|^{\gamma_1} \, |x_2|^{\gamma_2} |\hat{f}(x_1,x_2)|^{\beta} \, dx_1 \, dx_2 < \infty .$$

We have proved the theorem for the first quarter of the R^2 plane. We now define the following subsets for the second quarter of the R^2 plane, $n,k = 1,2 \ldots$

$$E_5 = \{ (x_1,x_2) : \; -2^{n} \leq x_1 \leq -2^{n-1} \; ; \; 2^{k-1} \leq x_2 \leq 2^{k} \}$$

$$E_6 = \{ (x_1,x_2) : \; -2^{n+1} \leq x_1 \leq -2^{-n} \; ; \; 2^{k-1} \leq x_2 \leq 2^{k} \}$$

(22)

$$E_7 = \{ (x_1,x_2) : \; -2^{n} \leq x_1 \leq -2^{n-1} ; \; 2^{-k} \leq x_2 \leq 2^{-k+1} \}$$

$$E_8 = \{ (x_1,x_2) : \; -2^{-n+1} \leq x_1 \leq -2^{-n} : 2^{-k} \leq x_2 \leq 2^{-k+1} \}$$

The variables x_1 and x_2 in (22) and in (12) are equal up to the sign, therefore the expressions $\sin^2 h_1x_1$, $|x_1|^{\gamma_1}$ $\sin^2 x_2h_2$ and $|x_2|^{\gamma_2}$ have the same estimate as those thus obtained. Hence the integrals

$$\int\limits_{-\infty}^{-1}\int\limits_{1}^{\infty}, \quad \int\limits_{-1}^{0}\int\limits_{1}^{\infty}, \quad \int\limits_{-\infty}^{-1}\int\limits_{0}^{1}, \quad \int\limits_{-1}^{0}\int\limits_{0}^{1}$$

coresponding to the double series in n and k on the subsets (22) have estimates up to a constant, which are equal to estimates of the integrals (15) - (19) . Hence

$$(23) \int\limits_{-\infty}^{0}\int\limits_{0}^{\infty} |x_1|^{\gamma_1} \, |x_2|^{\gamma_2} \left|\hat{f}(x_1,x_2)\right|^{\beta} dx_1 \, dx_2 < \infty.$$

Analogous results can be obtained for remaining parts of the R^2 plane. The theorem is proved.

Note. Denoting $P_n = \left(2^{na} + 2^{nc}\right)\left(2^{kb} + 2^{kd}\right) =$

$$= 2^{na}\left(1 + 2^{n(c-a)}\right) 2^{kb}\left(1 + 2^{k(d-b)}\right)$$

and putting $c = a$, $d = b$ we obtain $2\beta = 3\gamma_1 - 2$, $2\beta = 3\gamma_2 - 2$, hence $a = b = \beta$. Therefore $P_n = 4.2^{n+k}$.

Then, for the convergence of (10) it is sufficient to assume that

$$\sum_n \sum_k \left[\omega^{(1,2)}_{2}\left(2^{-n}, 2^{-k}\right)\right]^{\beta} 2^{(n+k)\beta} < \infty.$$

In particular, putting $\beta = 1$ we have $\gamma_1 = \gamma_2 = 0$ and the convergence of the series $\sum_n \sum_k \omega^{(1,2)}_{2}(2^{-n}, 2^{-k}) \cdot 2^{(n+k)}$

implies the convergence of the integral

$$\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \left|\hat{f}(x_1,x_2)\right| dx_1 \, dx_2 .$$

R e f e r e n c e s

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