On a modular equation II

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1. In [1], we investigated the modular equation of the form

$$x(t) = \varkappa\, \rho(t,x),$$

where $\rho(t,x)$ is a family of pseudomodulars depending on the parameter t, applying the Banach fix-point theorem. We investigated solutions belonging to the modular space X_{ρ_s}, where ρ_s is given by formula (2) below. Here, we shall continue the investigation of the operator

$$(1) \qquad (A(x))(t) = \varkappa\, \rho(t,x), \quad \varkappa \neq 0,$$

in a ball in X_{ρ_s}.

Let Ω be an abstract nonempty set, Σ - a σ- algebra of subsets of Ω, μ - a nonnegative, finite measure in Σ and $\mathcal{X}$ - the space of real-valued, Σ - measurable functions in Ω; two functions equal μ - almost everywhere on Ω are regarded as one and the same element of $\mathcal{X}$. We denote by $\mathcal{X}_0$ an arbitrary fixed linear subspace of $\mathcal{X}$. Let $\rho : \Omega \times \mathcal{X}_0 \to \langle 0, \infty)$ be a map of $\Omega \times \mathcal{X}_0$ in the extended real halfline, such that:

1) $\rho(t,x)$ is a convex pseudomodular in $\mathcal{X}_0$ for a.e. $t \in \Omega$,

2) if $\rho(t,x) = 0$ a.e. in Ω, then $x = 0$,

3) $\rho(t,x)$ is a $\sum$ - measurable function in the variable t in Ω, for every $x \in \mathcal{X}_0$.

Let X be the set of functions $x \in \mathcal{X}_0$ such that $\rho(t, \lambda x) \to 0$ as $\lambda \to 0$ a.e. in Ω , and

(2) $\rho_s(x) = \int_\Omega \rho(t,x)\, d\mu$,

then ρ_s is a convex modular in X, and

$$X_{\rho_s} = \left\{ x : x \in X, \quad \rho_s(\lambda x) \to 0 \text{ as } \lambda \to 0 \right\}$$

is a modular space, which is a normed linear space under the norm

$$\|x\|_{\rho_s} = \inf \left\{ \eta > 0 : \rho_s\left(\frac{x}{\eta}\right) \leq 1 \right\}.$$

It is easily seen that X_{ρ_s} consists exactly of such elements $x \in X$ for which $\rho_s(\lambda_0 x) < \infty$ for some $\lambda_0 > 0$ (for the definitions, see [1] , [2]).

2. In the following, we denote for any $0 < M < \infty$, by $K_{\rho_s}/M/$ the ball in X_{ρ_s} with centre at 0 and with radius M, with respect to the norm $\| \ \|_{\rho_s}$.

Theorem 1. Let $0 < M < \infty$, $0 < M' < \infty$. Let us suppose that for every $x \in X$ and for all λ satisfying the inequalities $0 < \lambda \leq \frac{1}{M}$, there holds the inequality

(3) $\rho(t, \lambda \varkappa_\rho(\cdot, x)) \leq \rho\left(t, \frac{\lambda M' x}{M}\right)$ a.e. in Ω .

Then the operator A defined by (1) maps $K_{\rho_s}(M)$ in $K_{\rho_s}(M')$.

Proof. Integrating (3), we get

$$\rho_s(\lambda A(x)) \leq \rho_s\left(\frac{\lambda M'}{M} x\right),$$

Hence, $x \in X_{\rho_s}$ implies $A(x) \in X_{\rho_s}$. Now, let $x \in K_{\rho_s}(M)$, and let us take $\lambda = \frac{1}{M'}$, then $\rho_s\left(\frac{A(x)}{M'}\right) \leq \rho_s\left(\frac{x}{M}\right)$.

Since $\|x\|_{\rho_s} \leq M$, this implies $\rho_s\left(\frac{x}{M}\right) \leq 1$. Consequently, $\rho_s\left(\frac{A(x)}{M'}\right) \leq 1$, and we get

$\|A(x)\|_{\rho_s} \leq M'$, i.e. $A(x) \in K_{\rho_s}(M')$.

Theorem 2. Let ρ satisfy the assumption (3) of Theorem 1 with $M' = M$. Moreover, let us suppose that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for arbitrary $\eta > 0$ and for all $x, y \in K_{\rho_s}(M)$ there holds the inequality

$$(4) \quad \int_\Omega \rho\left(t, \frac{\rho(\cdot,x) - \rho(\cdot,y)}{\eta\varepsilon}\right) d\mu \leq \int_\Omega \rho\left(t, \frac{x-y}{\varkappa\eta\delta}\right) d\mu.$$

Then the operator A maps $K_{\rho_s}(M)$ into itself continuously.

Proof. By Theorem 1, A maps $K_{\rho_s}(M)$ into itself. Now, we have

$$\left\|\frac{x-y}{\delta}\right\|_{\rho_s} = |\varkappa| \inf\left\{\eta > 0 : \int_\Omega \rho\left(t, \frac{x-y}{\varkappa\eta\delta}\right) d\mu \leq 1\right\},$$

$$\left\|\frac{A(x) - A(y)}{\varepsilon}\right\|_{\rho_s} = |\varkappa| \inf\left\{\eta > 0 : \int_\Omega \rho\left(t, \frac{\rho(\cdot,x) - \rho(\cdot,y)}{\eta\varepsilon}\right) d\mu \leq 1\right\}.$$

Hence, applying (4) we observe that

$$\left\|\frac{x-y}{\delta}\right\|_{\rho_s} \leq 1 \quad \text{implies} \quad \left\|\frac{A(x) - A(y)}{\varepsilon}\right\|_{\rho_s} \leq 1,$$

but this shows the continuity of A.

Similarly as in [1], Theorem 3, the following theorem may be proved:

Theorem 3. Let ϱ satsify the assumptions of Theorem 1. Moreover, let us suppose there exists a number $\alpha > 0$ such that

$$\int_{\Omega} \left(t, \frac{\varrho(\cdot,x) - \varrho(\cdot,y)}{\eta}\right) d\mu \leq \int_{\Omega} \left(t, \frac{x-y}{\frac{\varkappa\eta}{\alpha}}\right) d\mu$$

for all $x, y \in K_{\varrho_s}(M)$ and for every $\eta > 0$. Then the operator A satisfies the inequality

$$(5) \qquad \|A(x) - A(y)\|_{\varrho_s} \leq \alpha \|x - y\|_{\varrho_s}$$

for all $x, y \in K_{\varrho_s}(M)$.

3. Now, we shall apply the above theorems to the case of $\mathcal{H}_0 = \mathcal{H}$, and of a modular

$$(6) \quad \varrho(t,x) = \int_{\Omega} k(t,s \ |x(s)|)\, d\mu(s),$$

where $k : \Omega \times \Omega \times \langle 0,\infty) \to \langle 0,\infty)$ is a measurable function in $\Omega \times \Omega \times \langle 0,\infty$, $k(t,s,0) = 0$ a.e. in $\Omega \times \Omega$ and $k(t,s,u)$ is a continuous, convex function of the variable u for almost all $t,s \in \Omega$ (see [1] , formula (7)). We have then $X = \mathcal{H}$, Let us denote

$$k_1^{\lambda}(t,u,v) = \int_{\Omega} k\left[t,s,\lambda k(s,u,v)\right] d\mu(s)$$

and

$$\varrho_1(t,x) = \int_{\Omega} k_1^{\lambda}(t,s, |x(s)|)\, d\mu(s)$$

for $\lambda > 0$. Then the following theorem holds:

Theorem 4. Let $0 < M < \infty$, $0 < M' < \infty$. If the inequality

$$(7) \quad \varrho_1^{\lambda}(t,x) \leq |\Omega| \, \varrho\left(t, \lambda \frac{M'}{M} x\right) \text{ a.e. in } \Omega$$

holds for all λ such that $0 < \lambda \leq \frac{1}{M'}$, then the operator

(8) $\quad (A(x))(t) = \varkappa \int_{\Omega} k(t, s, |x(s)|)\, d\mu(s)$

maps $K_{\varrho_s}(M)$ in $K_{\varrho_s}(M')$ for each $\varkappa$ such that $|\varkappa| \leqslant \frac{1}{|\Omega|}$.

Proof. Let $x \in X_{\varrho_s}$, $0 < \lambda \leqslant \frac{1}{M}$. Since $|\varkappa| \cdot |\Omega| \leqslant 1$, applying Jensen inequality we have

$$\varrho(t, \lambda \varkappa \varrho(\cdot, x)) \leqslant$$

$$\leqslant \frac{1}{|\Omega|} \int_{\Omega} \left\{ \int_{\Omega} k\left[t, s, \lambda k(s, u, |x(u)|)\right] d\mu(u) \right\} d\mu(s) =$$

$$= \frac{1}{|\Omega|} \int_{\Omega} k_1^{\lambda}(t, u, |x(u)|)\, d\mu(u) = \frac{1}{|\Omega|} \varrho_1^{\lambda}(t, x) \leqslant \varrho(t, \lambda_M^{M'} x)$$

a.e. in Ω, by the assumption (7). Applying Theorem 1 we conclude that A maps $K_{\varrho_s}(M)$ in $K_{\varrho_s}(M')$.

Theorem 5. Let us suppose that the assumptions of Theorem 4 are satisfied with $M' = M$ and that for every $\beta > 0$ there exists a number $\gamma > 0$ such that

(9)
$$\int_{\Omega} \left\{ \frac{1}{|\Omega|} \int_{\Omega} k\left[t, u, \frac{1}{\beta} |k(u, v, |x(v)|) - k(u, v, |y(v)|)|\right] d\mu(v) \right\} d\mu(u) \leqslant$$

$$\leqslant \int_{\Omega} k\left[t, u, \frac{|x(u) - y(u)|}{\gamma}\right] d\mu(u)$$

for almost all $t \in \Omega$ and for arbitrary $x, y \in K_{\varrho_s}(M)$. Then the operator A defined by (8) maps $K_{\varrho_s}(M)$ into itself, continuously.

Proof. Let us take any $\eta > 0$ and let $\varepsilon > 0$ be arbitrary.

Taking $\beta = \frac{\eta\varepsilon}{|\Omega|}$ and choosing $\delta = \frac{\gamma}{\varkappa\eta}$, where γ is given by (9), we obtain

$$\frac{1}{|\Omega|}\int_\Omega k\left[t, u, \frac{|\Omega|}{\eta\varepsilon}\left|k(u,v,|x(v)|) - k(u,v,|y(v)|)\right|\right] d\mu(v) \leq$$
$$\leq k\left[t, u, \frac{|x(u)-y(u)|}{|\varkappa|\eta\delta}\right]$$

for almost all $(t,u) \in \Omega\times\Omega$. Hence, applying Jensen inequality, we get

$$\int_\Omega \rho\left(t, \frac{\rho(\cdot,x)-\rho(\cdot,y)}{\eta\varepsilon}\right) d\mu(t) \leq$$

$$\leq \int_\Omega\left\{\int_\Omega k\left[t, u, \frac{1}{|\Omega|}\int_\Omega \frac{|\Omega|}{\eta\varepsilon}\left|k(u,v,|x(v)|) - k(u,v,|y(v)|)\right| d\mu(v)\right] d\mu(u)\right\} d\mu(t) \leq$$

$$\leq \int_\Omega\left\{\frac{1}{|\Omega|}\int_\Omega\left(\int_\Omega k\left[t, u, \frac{|\Omega|}{\eta\varepsilon}\left|k(u,v,|x(v)|) - k(u,v,|y(v)|)\right|\right] d\mu(u)\right) d\mu(u)\right\} d\mu(v)$$

$$\leq \int_\Omega\left\{\int_\Omega \frac{1}{|\Omega|}\int_\Omega k\left[t, u, \frac{|\Omega|}{\eta\varepsilon}\left|k(u,v,|x(u)|) - k(u,v,|y(v)|)\right|\right] d\mu(v)\, d\mu(u)\right\} d\mu(t) \leq$$

$$\leq \int_\Omega\left\{\int_\Omega k\left[t, u, \frac{|x(u)-y(u)|}{|\varkappa|\eta\delta}\right] d\mu(u)\right\} d\mu(t) =$$

$$= \int_\Omega \rho\left(t, \frac{x-y}{\varkappa\eta\delta}\right) d\mu(t),$$

for almost every $t \in \Omega$. This proves the inequality (4). Applying Theorem 2, we conclude the continuity of the operator **A**.

Theorem 6. Let us suppose that the assumptions of Theorem 4 are satisfied and that for a fixed number $\alpha > 0$ there holds the inequality

$$\int_\Omega \left\{ \frac{1}{|\Omega|} \int_\Omega k\left[t, u, \frac{|\Omega|}{\eta} |k(u, v, |x(v)|) - k(u, v, |y(v)|)|\right] d\mu(v) \right\} d\mu(u) \leq$$
$$\leq \int_\Omega k\left[t, u, \frac{\alpha}{|\mathcal{K}|\eta} |x(u) - y(u)|\right] d\mu(u)$$

for all $\eta > 0$, almost all $t \in \Omega$ and for arbitrary $x, y \in K_{\varrho_S}(M)$. Then the operator A satisfies the inequality (5) for all $x, y \in K_{\varrho_S}(M)$.

Proof. Arguing as in the proof of Theorem 5, we get

$$\int_\Omega \varrho\left(t, \frac{\rho(\cdot, x) - \rho(\cdot, y)}{\eta}\right) d\mu(t) \leq$$

$$\leq \int_\Omega \left\{ \int_\Omega \frac{1}{|\Omega|} \int_\Omega k\left[t, u \frac{|\Omega|}{\eta} |k(u, v, |x(v)|) - k(u, v, |y(v)|)|\right] d\mu(v) d\mu(u) \right\} d\mu(t) \leq$$

$$\leq \int_\Omega \left\{ \int_\Omega k\left[t, u, \frac{\alpha}{|\mathcal{K}|\eta} |x(u) - y(u)|\right] d\mu(u) \right\} d\mu(t) =$$

$$= \int_\Omega \varrho\left(t, \frac{x - y}{\frac{|\mathcal{K}|\eta}{\alpha}}\right) d\mu(t)$$

for a.e. $t \in \Omega$. By Theorem 3, we conclude the inequality (5).

We proved in [1], Theorem 6, that if k satisfies the following condition:

(C) for every $\varepsilon > 0$, there holds $\int_\Omega k(t,s,\varepsilon)\,d\mu(t) > 0$ for a.e. $s \in \Omega$,

then the space X_{ϱ_s} is complete.

Hence applying the Banach fix-point theorem in $K_{\varrho_s}(M)$ to the operator A defined as above, and to the operator $B(x) = A(x) + y$, $y \in X_{\varrho_s}$ being fixed, we obtain easily the following result:

Theorem 7. Let us suppose the assumptions of Theorem 6 to be satisfied, and let k satisfy the condition (C). Then the integral equation

$$x(t) = \varkappa \int_\Omega k(t,u,|x(u)|)\,d\mu(u)$$

has only the trivial solution $x = 0$ in $K_{\varrho_s}(M)$. Moreover, if $0 < \vartheta < 1$ is fixed and if the assumptions of Theorem 4 are satisfied with $M' = \vartheta M$ and $y \in K_{\varrho_s}((1-\vartheta)M)$, then the integral equation

$$x(t) = \varkappa \int_\Omega k(t,u,|x(u)|)\,d\mu(u) + y(t)$$

has exactly one solution in $K_{\varrho_s}(M)$.

4. As an example of modular of the form (5), let us take $\Omega = \langle 0,1 \rangle$, μ = the Lebesgue measure,

(10) $k(t,s,u) = k_1(t,s) \cdot |u|$,

where k_1 is measurable in $\langle 0,1\rangle \times \langle 0,1\rangle$, $\int_0^1 k_1(t,s)\,dt > 0$ for almost all $s \in \langle 0,1\rangle$. Then we have

$$\rho(t,x) = \int_0^1 k_1(t,s)\,|x(s)|\,ds,$$

$$k_1^\lambda(t,u,v) = \lambda \int_0^1 k_1(t,s)\,k_1(s,u)\,ds \cdot |v|,$$

$$\rho_1^\lambda(t,x) = \lambda \int_0^1 \left\{ \int_0^1 k_1(t,s)\,k_1(s,u)\,ds \right\} |x(u)|\,du$$

for any $\lambda > 0$. Moreover, $x \in X_{\rho_S}$ is equivalent to the condition

$$\int_0^1 \int_0^1 k_1(t,s)\,|x(s)|\,ds\,dt < \infty$$

Supposing that

$$(11) \qquad \int_0^1 k_1(t,s)\,k_1(s,u)\,ds \leqslant \vartheta\, k_1(t,u)$$

for almost all $(t,u) \in \langle 0,1\rangle \times \langle 0,1\rangle$, $0 < \vartheta \leqslant 1$, we obtain the condition (7) with any $M' \leqslant \vartheta M$ for arbitrary $\lambda > 0$, $|\varkappa| \leqslant 1$. Moreover, applying (11) we obtain the inequality (9) with $\gamma = \frac{\beta}{\vartheta}$. Furthermore, if $0 < \alpha \leqslant |\varkappa|\vartheta$, then the operator A satisfies the inequality (5). Obviously, the condition (C) is satisfied, and so X_{ρ_S} is complete. Hence, applying Theorem 7, we obtain the following result:

Theorem 8. Let k and k_1 be defined by (10), and let us suppose (11) to be satisfied. Moreover, let $|\varkappa| < 1$. Then the integral equation $x(t) = \varkappa \int_0^1 k_1(t,u) \cdot |x(u)|\,du + y(t)$ has for any $y \in K_{\rho_S}((1-\vartheta)M)$ exactly one solution x in $K_{\varrho_s}(M)$.

References

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