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# Covering functions by countably many functions from some families

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**Abstract.** Let $\mathcal{A}$ be a nonempty family of functions from $\mathbb{R}$ to $\mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be strongly countably $\mathcal{A}$-function if there is a sequence $(f_n)$ of functions from $\mathcal{A}$ such that $\mathrm{Gr}(f) \subset \bigcup_n \mathrm{Gr}(f_n)$ ($\mathrm{Gr}(f)$ denotes the graph of $f$). If $\mathcal{A}$ is the family of all continuous functions, the strongly countable $\mathcal{A}$-functions are called strongly countably continuous and were investigated in [Z. Grande and A. Fatz-Grupka, On countably continuous functions, *Tatra Mt. Math. Publ.*, 28:57–63, 2004], [G. Horbaczewska, On strongly countably continuous functions, *Tatra Mt. Math. Publ.*, 42:81–86, 2009], and [T.A. Natkaniec, On additive countably continuous functions, *Publ. Math.*, 79(1–2):1–6, 2011].

In this article, we prove that the families $\mathcal{A}(\mathbb{R})$ of all strongly countably $\mathcal{A}$-functions are closed with respect to some operations in dependence of analogous properties of the families $\mathcal{A}$, and, in particular, we show some properties of strongly countably differentiable functions, strongly countably approximately continuous functions, and strongly countably quasi-continuous functions.

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*Keywords:* Lebesgue measurability, Baire property, first Baire class, approximate continuity, differentiability

## 1 Introduction

Let $\mathbb{R}$ be the set of all reals, and let $\mathcal{A}$ be a nonempty family of functions from $\mathbb{R}$ to $\mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be strongly countably $\mathcal{A}$-function if there is a sequence $(f_n)$ of functions from $\mathcal{A}$ such that $\mathrm{Gr}(f) \subset \bigcup_n \mathrm{Gr}(f_n)$ ($\mathrm{Gr}(f)$ denotes the graph of $f$). If $\mathcal{A}$ is the family of all continuous functions, the strongly countable $\mathcal{A}$-functions are called strongly countably continuous and were investigated in [4, 5, 7].

In this article, we prove that the families $\mathcal{A}(\mathbb{R})$ of all strongly countably $\mathcal{A}$-functions are closed with respect to some operations in dependence of analogous properties of the families $\mathcal{A}$. Moreover, we show some properties of strongly countably differentiable functions and strongly countably quasi-continuous functions. We recall that a function $f : \mathbb{R} \to \mathbb{R}$ is quasi-continuous at a point $x \in \mathbb{R}$ if, for each open interval $I$ containing $x$ and every real $r > 0$, there is an open interval $J \subset I$ such that $f(J) \subset (f(x) - r, f(x) + r)$ (see [6, 8]).

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## 2 The main results

### 2.1 The closedness of $\mathcal{A}(\mathbb{R})$ with respect to some operations

We start from the obvious observation that if a family $\mathcal{A}$ is contained in $\mathcal{E}$, then $\mathcal{A}(\mathbb{R}) \subset \mathcal{E}(\mathbb{R})$.

Let $F : \mathbb{R}^2 \to \mathbb{R}$. Theorems 1 and 2 are generalizations of theorems from [4], and their proofs are similar to those from [4].

**Theorem 1.** *Assume that the family $\mathcal{A}$ is closed with respect to the operation $F$, i.e., for arbitrary functions $\phi, \psi \in \mathcal{A}$, the function $F(\phi, \psi) \in \mathcal{A}$. Then, for arbitrary two functions $f, g \in \mathcal{A}(\mathbb{R})$, the function $F(f, g) \in \mathcal{A}(\mathbb{R})$.*

*Proof.* There are sequences $(f_n)$ and $(g_n)$ of functions from $\mathcal{A}$ such that

$$\mathrm{Gr}(f) \subset \bigcup_n \mathrm{Gr}(f_n) \quad \text{and} \quad \mathrm{Gr}(g) \subset \bigcup_n \mathrm{Gr}(g_n).$$

For $n = 1, 2, \ldots$, let

$$A_n = \bigl\{x;\ f_n(x) = f(x)\bigr\} \quad \text{and} \quad B_n = \bigl\{x;\ g_n(x) = g(x)\bigr\}.$$

Evidently,

$$\bigcup_n A_n = \bigcup_n B_n = \mathbb{R}.$$

For $n, k = 1, 2, \ldots$, put $D_{n,k} = A_n \cap B_k$. By our assumption the functions $F(f_n, g_k)$ belong to $\mathcal{A}$ for $n, k = 1, 2, \ldots$. Moreover, for $x \in D_{n,k}$, we have $F(f(x), g(x)) = F(f_n(x), g_k(x))$. Since

$$\mathbb{R} = \bigcup_n A_n = \bigcup_n B_n = \bigcup_{n,k \geqslant 1} D_{n,k},$$

we obtain that

$$\mathrm{Gr}\bigl(F(f, g)\bigr) \subset \bigcup_{n,k \geqslant 1} \mathrm{Gr}\bigl(F(f_n, g_k)\bigr),$$

and consequently, $F(f, g)$ belongs to $\mathcal{A}(\mathbb{R})$. $\square$

From the above Theorem 1 we immediately get the following:

**Corollary 1.** *Assume that the family $\mathcal{A}$ is closed with respect to the addition* (*subtraction*) [*multiplication by constant*] {*multiplication*}. *Then the family $\mathcal{A}(\mathbb{R})$ has the same property.*

For the investigation of the closedness of $\mathcal{A}(\mathbb{R})$ with respect to the division, we introduce the subfamily $\mathcal{A}_1(\mathbb{R}) \subset \mathcal{A}(\mathbb{R})$ such that $f \in \mathcal{A}_1(\mathbb{R})$ if and only if there is a sequence of functions $f_n \in \mathcal{A}$, $n = 1, 2, \ldots$, such that $\mathrm{Gr}(f) \subset \bigcup_{n \geqslant 1} \mathrm{Gr}(f_n)$ and $f_n(\mathbb{R}) \subset \mathbb{R} \setminus \{0\}$ for $n \geqslant 1$.

Observe that if $\mathcal{A}$ is the family of all continuous functions or the family of all differentiable functions, then every function $f \in \mathcal{A}(\mathbb{R})$ with $f(\mathbb{R}) \subset \mathbb{R} \setminus \{0\}$ belongs to $\mathcal{A}_1(\mathbb{R})$.

**Corollary 2.** *If, for arbitrary two functions $\phi, \psi \in \mathcal{A}$ with $\psi(\mathbb{R}) \subset \mathbb{R} \setminus \{0\}$, the quotient $\phi/\psi$ belongs to $\mathcal{A}$, then, for arbitrary two functions $f, g \in \mathcal{A}(\mathbb{R})$ with $g \in \mathcal{A}_1(\mathbb{R})$, the quotient $f/g$ belongs to $\mathcal{A}(\mathbb{R})$.*

*Proof.* It suffices to repeat the proof of Theorem 1 for the operation $F(x, y) = x/y$ defined on $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$. The hypothesis that $g \in \mathcal{A}_1(\mathbb{R})$ implies that there is a sequence of functions $g_n \in \mathcal{A}$ such that $g_n(\mathbb{R}) \subset \mathbb{R} \setminus \{0\}$ for $n \geqslant 1$ and $\mathrm{Gr}(g) \subset \bigcup_n \mathrm{Gr}(g_n)$. $\square$

Similarly to Theorem 1, we obtain the following:

**Corollary 3.** *Assume that the family $\mathcal{A}$ is closed with respect to the operation $F_1(x,y) = \max(x,y)$ ($F_2(x,y) = \min(x,y)$). Then the family $\mathcal{A}(\mathbb{R})$ has the same property.*

**Theorem 2.** *Assume that the family $\mathcal{A}$ is closed with respect to the superposition. Then the family $\mathcal{A}(\mathbb{R})$ has the same property.*

*Proof.* Fix two functions $f, g \in \mathcal{A}(\mathbb{R})$. There are sequences $(f_n)$ and $(g_n)$ of functions from $\mathcal{A}$ such that

$$\mathrm{Gr}(f) \subset \bigcup_n \mathrm{Gr}(f_n) \quad \text{and} \quad \mathrm{Gr}(g) \subset \bigcup_n \mathrm{Gr}(g_n).$$

For $n = 1, 2, \ldots$, let

$$A_n = \{x;\ f_n(x) = f(x)\} \quad \text{and} \quad B_n = \{x;\ g_n(x) = g(x)\}.$$

Then

$$\bigcup_n A_n = \bigcup_n B_n = \mathbb{R}.$$

For $n, k = 1, 2, \ldots$, put

$$C_{n,k} = (g_n)^{-1}(A_k) \cap B_n.$$

Observe that, for each point $x \in \mathbb{R}$, there is an index $n_1$ such that $g_{n_1}(x) = g(x)$. Next, there is an index $k_1$ with $g(x) \in A_{k_1}$, that is,

$$f\bigl(g(x)\bigr) = f_{k_1}\bigl(g(x)\bigr) = f_{k_1}\bigl(g_{n_1}(x)\bigr).$$

So,

$$x \in B_{n_1} \cap (g_{n_1})^{-1}(A_{k_1}) = C_{n_1,k_1} \quad \text{and} \quad \mathbb{R} = \bigcup_{n,k \geqslant 1} C_{n,k}.$$

If $x \in C_{n,k}$, then $f(g(x)) = f(g_n(x)) = f_k(g_n(x))$. So,

$$\mathrm{Gr}\bigl(f(g)\bigr) \subset \bigcup_{n,k \geqslant 1} \mathrm{Gr}\bigl(f_k(g_n)\bigr).$$

Since the compositions $f_k(g_n)$, $n, k \geqslant 1$, belong to $\mathcal{A}$, the proof is complete. $\square$

The above results when $\mathcal{A}$ is the family of all continuous functions were obtained in [4]. If $\mathcal{A}$ is the family of all constant functions, then, evidently, a function $f \in \mathcal{A}(\mathbb{R})$ (we will say that $f$ is strongly countably constant) if and only if the image $f(\mathbb{R})$ is countable. Now we will investigate some other cases.

### 2.2 Strongly countably differentiable functions

Let $\mathcal{A}$ be the family of all differentiable functions, which will be denoted by $\Delta$. This family is closed with respect to the sums, differences, products, quotients (if the image of the denominator is contained in $\mathbb{R} \setminus \{0\}$, and superpositions. It is not closed with respect to the operations max and min. Strongly countably $\Delta$-functions will be called strongly countably differentiable. The characteristic functions of nonmeasurable (in the sense of Lebesgue) sets are strongly countably constant and, thus, also strongly countably differentiable. Evidently, such functions are nonmeasurable (and, thus, also discontinuous).

**Theorem 3.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then $f \in \Delta(\mathbb{R})$ if and only if, for each nonempty closed set $H \subset \mathbb{R}$, there are an open interval $I$ such that $I \cap H \neq \emptyset$ and a differentiable function $g : \mathbb{R} \to \mathbb{R}$ such that $f|(H \cap I) = g|(H \cap I)$.*

*Proof.* *Necessity.* Let $f \in \Delta(\mathbb{R})$ be a continuous function, and let $H$ be a nonempty closed set. If $H$ contains isolated points, then the proof is evident. So, we can assume that $H$ is a nonempty perfect set. There is a sequence $(f_n)$ of differentiable functions such that

$$\mathrm{Gr}(f) \subset \bigcup_n \mathrm{Gr}(f_n).$$

Since $H$ is a complete space, there is an index $n$ such that the set $B = \{x \in H;\ f_n(x) = f(x)\}$ is of the second category in $H$. But $f$ and $f_n$ are continuous, so the set $B$ is closed. So, there is an open interval $I$ such that $\emptyset \neq I \cap H \subset B$. To finish the proof of the necessity, it suffices to put $g_H = f_n$.

*Sufficiency.* For $H = \mathbb{R}$, we find an open interval $I_0$ and a differentiable function $f_0$ such that $f/I_0 = f_0/I_0$. Next, by transfinite induction we define a transfinite sequence of open intervals $I_\alpha$, $\alpha < \gamma$, with rational endpoints such that

$$E_\alpha = I_\alpha \cap \left(\mathbb{R} \setminus \bigcup_{\beta<\alpha} I_\beta\right) \neq \emptyset,$$

and, for the restricted functions $f/E_\alpha$, there are differentiable functions $f_\alpha : \mathbb{R} \to \mathbb{R}$ with $f_\alpha/E_\alpha = f/E_\alpha$. Since $\gamma$ is a countable ordinal, the proof of the sufficiency is complete. □

*Remark 1.* The referee observed that the same proof of Theorem 3 works if the hypothesis of the continuity of the function $f$ is replaced by $f \in B_1^*$. Recall that $f \in B_1^*$ if, for each nonempty closed set $E$, there is an open interval $I$ with $I \cap E \neq \emptyset$, and the restriction $f/(I \cap E)$ is continuous (see [9]). Moreover, he observed a possibility of some generalization of Theorem 3 for some subfamilies $\mathcal{A} \subset B_1^*$ instead of $\Delta$.

**Corollary 4.** *If $f \in \Delta(\mathbb{R}) \cap B_1^*$, then it is differentiable on an open dense set.*

**Corollary 5.** *If a continuous function $f : \mathbb{R} \to \mathbb{R}$ is nowhere differentiable, then it is not strongly countably differentiable.*

*Remark 2.* The referee informed me that the existence of a continuous function that cannot be covered by countably many differentiable functions follows from theorem of Morayne (see Theorem 4.4 in [3]).

### 2.3 Strongly countably approximately continuous functions

We recall that the family of all measurable subsets $A \subset \mathbb{R}$ such that each point $x \in A$ is a density point of $A$, i.e., $\lim_{h\to 0^+} m(A \cap [x-h, x+h])/(2h) = 1$, where $m$ denotes the Lebesgue measure in $\mathbb{R}$, is the so-called density topology (compare [1]). Denote by $T_d$ the density topology and by $T_e$ the natural topology in $\mathbb{R}$. The continuity of functions from $(\mathbb{R}, T_d)$ to $(\mathbb{R}, T_e)$ is said to be the approximate continuity (see [1]). It is known that there are approximately continuous functions that are not strongly countably continuous (see [4]). In this section, we investigate strong countable approximate continuity. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be strongly countably approximately continuous if there are approximately continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ such that $\mathrm{Gr}(f) \subset \bigcup_n \mathrm{Gr}(f_n)$. The approximate continuity of a function $f : \mathbb{R} \to \mathbb{R}$ implies the measurability (in the sense of Lebesgue) of its graph. So, the graphs of strongly countably approximately continuous functions are measurable.

**Theorem 4.** *There are measurable functions $f : \mathbb{R} \to \mathbb{R}$ that are not strongly countably approximately continuous.*

*Proof.* Let $A \subset [0,1]$ be a nowhere dense compact set of positive measure. The difference $B$ of $A$ and the union of all open intervals $I$ with rational endpoints and $m(A \cap I) = 0$ is a compact set such that $m(B \cap J) > 0$ for each open interval $J$ with $B \cap J \neq \emptyset$. Enumerate in a transfinite sequence $(C_\alpha)_{\alpha<\omega_c}$ ($\omega_c$ denotes the first ordinal of continuum cardinality) the family of all closed subsets of $B \times \mathbb{R}$ that are of positive plane measure (in the sense of Lebesgue). Next, by transfinite induction, we find a transfinite sequence of points $(x_\alpha, y_\alpha) \in C_\alpha$, $\alpha < \omega_c$, such that $x_\alpha \neq x_\beta$ for $\alpha \neq \beta$, $\alpha, \beta < \omega_c$. Let

$$g(x_\alpha) = y_\alpha \quad \text{for } \alpha < \omega_c \quad \text{and} \quad g(x) = 0 \quad \text{otherwise on } \mathbb{R}.$$

Then the graph of the restricted function $g|B$ is not measurable in $B \times \mathbb{R}$. Since $B$ is a nowhere dense closed set, there is (see [10, Thm. 13.1]) an autohomeomorphism $h$ of $[0,1]$ such that $h([0,1]) = [0,1]$ and $m(h^{-1}(B)) = 0$. Moreover, let

$$f(x) = g\big(h(x)\big) \quad \text{for } x \in h^{-1}(B) \quad \text{and} \quad f(x) = 0 \quad \text{otherwise on } \mathbb{R}.$$

Since the values of the function $f$ are equal 0 for $x \in \mathbb{R} \setminus h^{-1}(B)$, it is measurable. We will prove that $f$ is not strongly countably approximately continuous. Suppose that, contrary to our claim, there is a sequence of approximately continuous functions $f_n : \mathbb{R} \to \mathbb{R}$, $n = 1, 2, \ldots$, such that $\mathrm{Gr}(f) \subset \bigcup_n \mathrm{Gr}(f_n)$. Since $f_n$, $n \geqslant 1$, are of the first Baire class, the functions $\phi_n(x) = f_n(h^{-1}(x))$ for $x \in \mathbb{R}$ have the same property, and consequently, their graphs are of plane measure 0. But $\mathrm{Gr}(g/B) \subset \bigcup_n \mathrm{Gr}(\phi_n)$, so it is of plane measure 0. This contradicts the fact that the graph $\mathrm{Gr}(g/B)$ is not measurable. □

*Remark 3.* Observe that, for each countable family of Borel functions $\psi_n : \mathbb{R} \to \mathbb{R}$, $n \geqslant 1$, and for the function $f$ from the proof of Theorem 4, we have $\mathrm{Gr}(f) \setminus \bigcup_n \mathrm{Gr}(\psi_n) \neq \emptyset$.

**Theorem 5.** *If a function $f : \mathbb{R} \to \mathbb{R}$ is of the first Baire class, then it is strongly countably approximately continuous.*

*Proof.* By Lusin's theorem, for each positive integer $n \geqslant 2$, there is a closed set $A_n \subset [-n, n]$ such that the restriction $f|A_n$ is continuous and $m([-n,n] \setminus A_n) < 1/n$. For each $n \geqslant 2$, there is a continuous function (so also approximately continuous) $f_n : \mathbb{R} \to \mathbb{R}$ such that $f_n/A_n = f/A_n$. Let $B = \mathbb{R} \setminus \bigcup_n A_n$. Observe that $m(B) = 0$. Indeed, suppose that, contrary to our claim, there is a positive real $r$ with $m(B) > r$. Since $m(B) = \lim_{n\to\infty} m([-n,n] \setminus \bigcup_{k \leqslant n} A_k)$, there is an index $k \geqslant 2$ with $m([-n,n] \setminus \bigcup_{i \leqslant n} A_i) > r$ for $n \geqslant k$. But this contradicts the inequalities $m([-n,n] \setminus \bigcup_{i \leqslant n} A_i) \leqslant m([-n,n] \setminus A_n) < 1/n$ for $n \geqslant 2$. So, $m(B) = 0$. Since $f$ is of the first Baire class and $m(B) = 0$, by Laczkovich and Petruska theorem from [7] there is an approximately continuous function $f_1 : \mathbb{R} \to \mathbb{R}$ such that $f_1/B = f/B$. The evident inclusion $\mathrm{Gr}(f) \subset \bigcup_n \mathrm{Gr}(f_n)$ completes the proof. □

The following problem is natural.

*Problem 1.* Does there exist functions $f : \mathbb{R} \to \mathbb{R}$ of the second Baire class that are not strongly countably approximately continuous?

The referee informed me that the positive answer to this problem follows from Corollary 3.4 in [2] (see also [3], the foot of page 160), where the authors proved that, for each ordinal $\alpha < \omega_1$, there is a function $f \in B_{\alpha+1}$ for which there is no countable partition $\{X_n; n \in \omega\}$ of $\mathbb{R}$ such that $f/X_n \in B_\alpha(X_n)$. Clearly, such a function $f$ cannot be covered by countably many functions from the class $B_\alpha$. This result implies also Theorem 4 and Remark 3.

### 2.4 Strongly countably quasi-continuous functions

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be strongly countably quasi-continuous if there is a sequence of quasi-continuous functions $f_n : \mathbb{R} \to \mathbb{R}$, $n = 1, 2, \ldots$, such that $\mathrm{Gr}(f) \subset \bigcup_n \mathrm{Gr}(f_n)$. Observe that the graphs of strongly countably quasi-continuous functions are of the first category on the plane $\mathbb{R}^2$ and there are strongly countably quasi-continuous functions that do not have the Baire property, for example, the characteristic functions of sets without the Baire property.

The main theorem in this section is the following.

**Theorem 6.** *If a function $f : \mathbb{R} \to \mathbb{R}$ has the Baire property, then it is strongly countably quasi-continuous.*

*Proof.* Without loss of generality, we can assume that $f$ is bounded; otherwise, we can consider the functions $\psi_n(x) = f(x)$ if $|f(x)| \leqslant n$ and $\psi_n(x) = 0$ otherwise on $\mathbb{R}$ and apply the observation that $\mathrm{Gr}(f) \subset \bigcup_n \mathrm{Gr}(\psi_n)$.

Since $f$ has the Baire property, there is an $F_\sigma$-set $A$ of the first category such that the restriction $f|(\mathbb{R} \setminus A)$ is continuous. Put $f_1(x) = f(x)$ for $x \in \mathbb{R} \setminus A$ and $f_1(x) = \limsup_{t \in \mathbb{R} \setminus A,\, t \to x} f(t)$ for $x \in A$. Observe that $f_1$ is a quasi-continuous function.

Since $A$ is an $F_\sigma$-set of the first category, there are nowhere dense closed sets $A_n$, $n \geqslant 2$, such that $A = \bigcup_{n \geqslant 2} A_n$. Now fix an integer $n \geqslant 2$ and a component $I$ of the set $\mathbb{R} \setminus A_n$. If $x$ is the right endpoint of the component $I$, then there are points $a_k(I) \in I$ such that $a_k(I) < a_{k+1}(I)$ for $k \geqslant 1$ and $\lim_{k \to \infty} a_k(I) = x$. Similarly, if $y$ is the left endpoint of the component $I$, there are points $b_k(I) \in I$ such that $a_1(I) > b_1(I) > b_2(I) > \cdots$ and $\lim_{k \to \infty} b_k(I) = y$. For this component $I$, we define a continuous function $\phi_I : I \to \mathbb{R}$ such that $\phi_I([a_{2k}(I), a_{2k+1}(I)]) = \phi_I([b_{2k+1}(I), b_{2k}(I)]) = [-k, k]$ for $k \geqslant 1$. Now let $f_n(x) = f(x)$ for $x \in A_n$ and $f_n(x) = \phi_I(x)$ on the components $I$ of the complement $\mathbb{R} \setminus A_n$. Evidently, the functions $f_n$, $n \geqslant 2$, are quasi-continuous and $\mathrm{Gr}(f) \subset \bigcup_{n \geqslant 1} \mathrm{Gr}(f_n)$. $\square$

## 3 Final problem

We have observed that all functions $f : \mathbb{R} \to \mathbb{R}$ of the first Baire class are strongly countably approximately continuous and strongly countably quasi-continuous.

*Problem 2.* Let $f : \mathbb{R} \to \mathbb{R}$ be a function of Baire 1 class. Do there exist a sequence of approximately continuous and, simultaneously, quasi-continuous functions $f_n : \mathbb{R} \to \mathbb{R}$, $n = 1, 2, \ldots$, such that $\mathrm{Gr}(f) \subset \bigcup_n \mathrm{Gr}(f_n)$.

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