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ON THE MEASURABILITY OF FUNCTIONS WITH QUASI-CONTINUOUS AND UPPER SEMI-CONTINUOUS VERTICAL SECTIONS

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ABSTRACT. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a function with upper semicontinuous and quasi-continuous vertical sections $f_x(t) = f(x,t)$, $t, x \in \mathbb{R}$. It is proved that if the horizontal sections $f^y(t) = f(t,y)$, $y, t \in \mathbb{R}$, are of Baire class α (resp. Lebesgue measurable) [resp. with the Baire property] then f is of Baire class $\alpha + 2$ (resp. Lebesgue measurable and sup-measurable) [resp. has Baire property].

1. Introduction

Let $\mathbb{R}$ be the set of all reals. It is well known that there is a nonmeasurable (in the sense of Lebesgue) set $A \subset \mathbb{R}^2$ which intersects every straight line at most two points (see Sierpiński [8]). The horizontal sections $f^y(t) = f(t,y)$, $t, y \in \mathbb{R}$, and the vertical sections $f_x(t) = f(x,t)$, $t, x \in \mathbb{R}$, of the characteristic function $f = \kappa_A$ of the set A are upper semi-continuous everywhere and discontinuous at most two points. Nevertheless the characteristic function f of the set A is not Lebesgue measurable. In this article we prove that the simultaneous upper semi-continuity and quasi-continuuity of the sections f_x, $x \in \mathbb{R}$, and the measurability (Borel, Lebesgue or Baire) of the sections f^y, $y \in \mathbb{R}$, guarantee similar measurability of f.

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2. The main results

For the formulation of the main results of this note recall that a function $g\colon \mathbb{R} \to \mathbb{R}$ is quasicontinuos at a point $x \in \mathbb{R}$ if for each $\eta > 0$ there is a non-empty open interval $I \subset (x-\eta, x+\eta)$ such that $g(I) \subset \big(g(x)-\eta, g(x)+\eta\big)$ (see [4, 7]).

I. Measurabilities

Theorem 2.1. *Assume that the vertical sections f_x, $x \in \mathbb{R}$, of a bounded function $f\colon \mathbb{R}^2 \to \mathbb{R}$ are quasicontinuous and upper semicontinuous and there is a dense set $A \subset \mathbb{R}$ such that the horizontal sections f^y, $y \in A$, are of Baire class $\alpha \geq 0$. Then f is of Baire class $\alpha + 2$.*

Proof. Without loss of generality we can assume that the set A is countable. For all positive integers $n \geq 1$ and all integers $k \in \mathcal{Z}$ we find open intervals $I_{n,k} = (a_{n,k}, a_{n,k+1})$ with endpoints belonging to A such that $0 < a_{n,k+1} - a_{n,k} < \frac{1}{n}$, $a_{n,k} \in \big\{a_{n+1,i} : i \in \mathcal{Z}\big\}$ for $k \in \mathcal{Z}$ and $\lim\limits_{k\to-\infty} a_{n,k} = -\infty$ and $\lim\limits_{k\to\infty} a_{n,k} = \infty$. For $n \geq 1$ and $y \in I_{n,k}$ we put $f_n(x,y) = \sup\limits_{t\in I_{n,k}} f(x,t)$ and $f_n(x, a_{,n,k}) = f(x, a_{,n,k})$ for $k \in \mathcal{Z}$. We will prove that $f = \lim\limits_{n\to\infty} f_n$. Fix a point $(x,y) \in \mathbb{R}^2$ and a real $\eta > 0$. If $y = a_{n,k}$ for some $n \geq 1$ and $k \in \mathcal{Z}$ then evidently for all $w \in \mathbb{R}$ and $i \geq n$ we have $f_i(w,y) = f(w,y)$. Therefore we can assume that $y \neq a_{n,k}$ for all $n \geq 1$ and $k \in \mathcal{Z}$. Since the section f_x is upper semi-continuous, there is a real $\delta > 0$ such that $f(x,t) - f(x,y) < \frac{\eta}{2}$ for $t \in (y-\delta, y+\delta)$. Let m be a positive integer such that $\frac{1}{n} < \delta$ for $n \geq m$. Let $i \geq m$ be an integer. There is a unique integer $k(i)$ with $y \in I_{i,k(i)}$. Since $a_{i,k(i)}$ and $a_{i,k(i)+1}$ belong to $(y-\delta, y+\delta)$, we have $f(x,y) \leq \sup\limits_{t\in I_{i,k(i)}} f(x,t) = f_i(x,y) < f(x,y) + \eta$. So $|f_i(x,y) - f(x,y)| < \eta$ for $i \geq m$ and $\lim\limits_{n\to\infty} f_n = f$. Now we shall prove that the functions f_n are of Baire class $\alpha + 1$. Indeed, since the vertical sections f_x are quasi-continuous, for a fixed $a \in \mathbb{R}$ and integers n, k, the set

$$B_{n,k} = \big\{(x,y) \in \mathbb{R}\times I_{n,k} : \ f_n(x,y) > a\big\} = \bigcup_{y\in A\cap I_{n,k}} \big(\big\{x \in \mathbb{R} : \ f(x,y) > a\big\} \times I_{n,k}\big)$$

is in $\sum\limits_{\alpha+1}^{0}$ class whenever $\alpha > 0$. Therefore for $\alpha > 0$ the set

$$\big\{(x,y) \in \mathbb{R}^2 : \ f_n(x,y) > a\big\} = \bigcup_{k=-\infty}^{\infty} B_{n,k} \cup \bigcup_{k=-\infty}^{\infty} \big(\big\{x : \ f(x, a_{n,k}) > a\big\} \times \{a_{n,k}\}\big)$$

is in $\sum\limits_{\alpha+1}^{0}$ class and the functions f_n, $n \geq 1$, are of Baire class $\alpha + 1$. In the case, where $\alpha = 0$, we observe that the sets $B_{n,k}$, $k \in \mathcal{Z}$, are open, and consequently the restricted functions $f_n/B_{n,k}$ are of the first Baire class. This suffices to the relation that f_n is of Baire 1 class. □

Corollary 2.2. *Assume that the vertical sections f_x, $x \in \mathbb{R}$, of a bounded function $f\colon \mathbb{R}^2 \to \mathbb{R}$ are quasi-continuous and upper semi-continuous and there is a dense set $A \subset \mathbb{R}$ such that the horizontal sections f^y, $y \in A$, are Borel. Then f is a Borel function.*

P r o o f. Without loss of generality we can assume that the set A is countable and find a countable ordinal α such that the sections f^y, $y \in A$, are of Baire class α. By Theorem 2.1 the function f is of Baire class $\alpha + 2$. □

Example 2.3. Let $C \subset [0,1]$ be the ternary Cantor set and let (I_n) be an enumeration of all components of the set $[0,1] \setminus C$ such that $I_n \cap I_m = \emptyset$ for $n \neq m$. Put $I_n = (a_n, b_n)$ for $n \geq 1$ and find reals $c_n \in (a_n, b_n)$ and $r_n \in (0, b_n - a_n)$ such that $c_n > b_n - r_n$ and a_k is not in $(b_n, b_n + r_n)$ for $k < n$. Let T_n be the triangle with the vertices (c_n, c_n), $(b_n - r_n, b_n)$ and $(b_n + r_n, b_n)$. There is a continuous function $f_n\colon T_n \to [0,1]$ such that $f_n(b_n, b_n) = 1$ and $f_n(x,y) = 0$ for all points (x,y) belonging to the boundary of T_n with $y < b_n$. Let $g(x,y) = f_n(x,y)$ for $(x,y) \in T_n$, $n \geq 1$, and let $g(x,y) = 0$ otherwise on $\mathbb{R}^2$. Observe that all horizontal sections g^y are continuous and all vertical sections g_x are quasi-continuous. Moreover, if a section g_u is not upper semi-continuous at a point $w \in \mathbb{R}$ then $w \in C$. For such a point (u,w) we put $f(u,w) = \limsup\limits_{y \to w} g(x,y)$ and let $f(x,y) = g(x,y)$ for all other points $(x,y) \in \mathbb{R}^2$. Then the vertical sections f_x, $x \in \mathbb{R}$, are quasi-continuous and upper semi-continuous and for all $y \in \mathbb{R} \setminus C$ the sections f^y are continuous. Since $f(b_n, b_n) = 1$ and $f(a_n, a_n) = 0$ for $n \geq 1$, the function f is not of Baire 1 class. Therefore it is not true that if for the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ having upper semi-continuous and simultaneously quasi-continuous its vertical sections f_x, $x \in \mathbb{R}$, there is a dense set $A \subset \mathbb{R}$ such that the horizontal sections f^y, $y \in A$, are continuous, then f is of Baire 1 class. So in Theorem 2.1 the Baire class $\alpha + 2$ of f cannot be replaced by a smaller one.

Analogously as Theorem 2.1 we can prove the following theorem.

Theorem 2.4. *Assume that the vertical sections f_x, $x \in \mathbb{R}$, of a bounded function $f\colon \mathbb{R}^2 \to \mathbb{R}$ are quasi-continuous and upper semi-continuous and there is a dense set $A \subset \mathbb{R}$ such that the horizontal sections f^y, $y \in A$, are Lebesgue measurable (resp. have the Baire property). Then f is Lebesgue measurable (resp. has the Baire property).*

Remark 1. In Theorems 2.1 and 2.4 the assumption of the boundesness of f is not important. Indeed, we can investigate the function $g = \arctan(f)$ and apply the equality $f = \tan(g)$.

Observe that Theorem 2.4 is more general than [3: Theorem 5].

Remark 2. Martin Axiom implies that there is a Lebesgue nonmeasurable function $f\colon \mathbb{R}^2 \to [0,1]$ with upper semi-continuous and approximately continuous sections f_x, $x \in \mathbb{R}$, and Lebesgue measurable sections f^y, $y \in \mathbb{R}$, (see [2] and [1] for the information about the approximate continuity). In [5] M. Laczkovich and A. Miller present very interesting results concerning to the measurabilities (of Borel and Lebesgue) of functions of two variables with approximately continuous vertical sections.

Remark 3. It is known that the quasi-continuity everywhere and the continuity almost everywhere of all vertical sections f_x, $x \in \mathbb{R}$, of a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ and the Lebesgue measurability of all horizontal sections f^y, $y \in \mathbb{R}$, imply the Lebesgue measurability of f ([6]). The following example shows that there are upper semi-continuous and simultaneously qusi-continuous functions $g\colon \mathbb{R} \to \mathbb{R}$ which are not almost everywhere continuous.

Example 2.5. Let $C \subset [0,1]$ be a Cantor set of positive Lebesgue measure and let (I_n) be a sequence of all components of the set $[0,1] \setminus C$ with $I_n \neq I_m$ for $n \neq m$. For $n \geq 1$ let $I_n = (a_n, b_n)$ and $c_n = \frac{a_n+b_n}{2}$. Then the function

$$g(x) = \begin{cases} \;\;0 & \text{for } \; x \in C \cup (-\infty, 0] \cup [1, \infty) \\ -1 & \text{for } \; x = c_n, \;\; n \geq 1 \\ \text{linear} & \text{on the intervals } \; [a_n, c_n], \;\; [c_n, b_n], \;\; n \geq 1 \end{cases}$$

satisfies all requirements.

II. Sup-measurability

Recall that a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is said (L)-sup-measurable (resp. (B)-sup-measurable) if for each Lebesgue measurable (resp. with the Baire property) function $g\colon \mathbb{R} \to \mathbb{R}$ the Carathéodory superposition $h(x) = f(x, g(x))$ is Lebesgue measurable (resp. has the Baire property) (compare [10]).

Theorem 2.6. *Assume that the vertical sections f_x, $x \in \mathbb{R}$, of a bounded function $f\colon \mathbb{R}^2 \to \mathbb{R}$ are quasi-continuous and upper semi-continuous and there is a dense countable set $A \subset \mathbb{R}$ such that the horizontal sections f^y, $y \in A$, are Lebesgue measurable (resp. have the Baire property). Then f is (L)-sup-measurable (resp. (B)-sup-measurable).*

P r o o f. The same as in the proof of Theorem 2.1 for all positive integers $n \geq 1$ and all integers $k \in \mathcal{Z}$ we find open intervals $I_{n,k} = (a_{n,k}, a_{n,k+1})$ with endpoints belonging to A such that $0 < a_{n,k+1} - a_{n,k} < \frac{1}{n}$, $a_{n,k} \in \{a_{n+1,i} :\ i \in \mathcal{Z}\}$ for $k \in \mathcal{Z}$ and $\lim_{k\to-\infty} a_{n,k} = -\infty$ and $\lim_{k\to\infty} a_{n,k} = \infty$.

For $n \geq 1$ and $y \in I_{n,k}$ we put $f_n(x, y) = \sup_{t\in I_{n,k}} f(x, t)$ and $f_n(x, a_{n,k}) = f(x, a_{,n,k})$ for $k \in \mathcal{Z}$. Then $f = \lim_{n\to\infty} f_n$. Fix a Lebesgue measurable function $g\colon \mathbb{R} \to \mathbb{R}$. Since $f(x, g(x)) = \lim_{n\to\infty} f_n(x, g(x))$ for $x \in \mathbb{R}$, for the proof of the Lebesgue measurablity of the superposition $h(x) = f(x, g(x))$, it suffices to show that the superpositions $h_n(x) = f_n(x, g(x))$ are Lebesgue measurable for $n \geq 1$. For this we will prove that they are approximately continuous almost everywhere. The sets $E_{n,k} = \{x \in \mathbb{R} :\ g(x) = a_{n,k}\}$, $k \in \mathcal{Z}$, are Lebesgue measurable and the sections $f^{a_{n,k}}$ are Lebesgue measurable. Since Lebesgue measurable functions are approximately continuous almost everywhere, the restrictions $h_n/E_{n,k}(x) = f(x, a_{n,k})$, $k \in \mathcal{Z}$, of the superposition h_n are aproximately continuous almost everywhere. Similarly the sets $H_{n,k} = \{x \in \mathbb{R} :\ g(x) \in (a_{n,k}, a_{n,k+1})\}$ are Lebesgue measurable for $k \in \mathcal{Z}$. The restrictions $g/H_{n,k}$ are approximately continuous almost everywhere. Let $w \in (a_{n,k}, a_{n,k+1})$. Observe that $f_n(x, y) = f_n(x, w)$ for $y \in (a_{n,k}, a_{n,k+1})$ and $x \in \mathbb{R}$. Since f_n is Lebesgue measurable by Theorem 2.4, the section $(f_n)^w$ is also Lebesgue measurable. As Lebesgue measurable the section $(f_n)^w$ is approximately continuous almost everywhere. If $x \in H_{n,k}$ is a density point of the set $H_{n,k}$ at which $(f_n)^w$ is aproximately continuous then from the equality $f_n(t, g(t)) = f_n(t, w)$ for $t \in H_{n,k}$ it follows that the function $h_n(t) = f_n(t, g(t))$ is also approximately continuous at x. This finishes the proof of (L)-sup-measurability of f. For the proof of the (B)-sup-measurability of f we take a function $g\colon \mathbb{R} \to \mathbb{R}$ with the Baire property and consider the superposition $h_n(x) = f_n(x, g_n(x)$. The restrictions $h_n/E_{n,k}$, $k \in \mathcal{Z}$, have evidently the Baire property. Next finding $w \in (a_{n,k}, a_{n,k+1})$ we observe that there is a residual set K such that the restrictions g/K and $(f_n)^w/K$ are continuous. Therefore the restrictions $h_n/(K \cap H_{n,k})$ are continuous and h_n has the Baire property. This finishes the proof of (B)-sup-measurability of f. □

3. Final problem

In the theory of differential equations it is well known that if a locally bounded functions $f\colon D \to \mathbb{R}$, where $D \subset \mathbb{R}^2$ is an open set containing a point (x_0, y_0), is such that the sections f_x, $x \in Pr_X(D)$ (the projection of D), are continuous and the sections f^y, $y \in Pr_Y(D)$, are Lebesgue measurable, then there is a Carathéodory's solution of the Cauchy problem $y'(x) = f(x, y(x))$, with the initial condition $y(x_0) = y_0$, i.e. y is an absolutely continuous function satisfying almost everywhere in its domain the equation $y'(x) = f(x, y(x))$ and such that $y(x_0) = y_0$.

Problem 3.1. *Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a locally bounded function having upper semi-continuous and simultaneously quasi-continuous vertical sections f_x, $x \in \mathbb{R}$. Assume that there is a dense set $A \subset \mathbb{R}$ such that the sections f^y, $y \in A$, are Lebesgue measurable. Does there exist a Carathéodory's solution of Cauchy's problem $y'(x) = f(x, y(x))$ with the initial condition $y(x_0) = y_0$?*

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