

# MATHEMATICAL MODELING OF RANDOM DIFFUSION FLOWS IN TWO-PHASE MULTILAYERED STOCHASTICALLY NONHOMOGENEOUS BODIES

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**Abstract:** An approach for studying stochastic diffusion flows of admixture particles in bodies of multiphase randomly nonhomogeneous structures is proposed, according to which initial-boundary value problems of diffusion are formulated for flow functions and methods of solution construction are adapted for the formulated problems. By this approach the admixture diffusion flow is investigated in a two-phase multilayered strip for the uniform distribution of phases under conditions of constant flow on the upper surface and zero concentration of admixture on the lower surface. An integro-differential equation equivalent to the original initial-boundary value problem is constructed. Its solution is found in terms of the Neumann series. Calculation formulae are obtained for the diffusion flow averaged over the ensemble of phase configurations under both zero and constant nonzero initial concentrations. Software is developed, a dependence of averaged diffusion flows on the medium characteristics is studied and general regularities of this process are established.

**Keywords:** diffusion process, mass flow, random structure, Neumann series, averaging over the ensemble of phase configurations

## 1. Introduction

In the study of mass transfer processes in porous media, complex geological structures, composite materials, nanostructures, *etc.*, an important characteristic

of the process, in addition to the concentration of migrating substance and chemical potentials, is the diffusion flux. Diffusion fluxes have an important value at the investigation of parameters of membranes and filters, material diagnostics (diffusion-structural analysis), determining the structure of metal growth during the solidification of alloys, *etc.* [1–5]. In the existent industrial systems of treatment of drinkable water and polluted drains multilayered filters with different porosity of layers are widely used. Their efficiency depends strongly on the porosity as well as the corresponding geometrical parameters. As a rule, in the engineering practice simulation to calculate such filters is used [6] solving non-linear problems of filtration of sewage by numerical methods.

On modeling admixture mass fluxes in multiphase bodies of a randomly nonhomogeneous structure there are significant difficulties during averaging over an ensemble of phase configurations because the correlation functions between the gradient of the stochastic field of the concentration and random diffusion coefficient are unknown. To solve this problem some authors [7, 8] propose balance equations for porous bodies to construct homogenized media, the physical characteristics of which are the averaged magnitudes taking into account the difference between coefficients of the phases and herewith interaction of phases is neglected. In the works [9, 10] the processes of heat transfer and diffusion in one-dimensional periodical stratified structures are studied on the basis of the method of homogenization. Here the micro-macro approach to description of physical processes is used, wherefore they impose a constraint of admissibility of description for the specific heterogeneous material by an equivalent homogeneous medium. In the paper [11] the random flow is determined after the Darcy law with the filtration coefficient being a function of spatial coordinate. The methods of both small perturbations and smoothing (with the corresponding constraints) are applied to construct the problem solution. And also they impose the condition of normal distribution of phases, which makes it impossible to determine the averaged mass flow, so the author defines the two-point function of covariation only.

In the work [12] the original approach is proposed, according to which the diffusion equation for the function of mass flux is constructed and initial-boundary value problems are formulated directly to the flux. However, within the scope of such approach it is necessary to formulate reasonable initial and boundary conditions, as in the case where values of the flux on the “top” body boundary are much greater than on the “bottom” one, an unlimited amount of the diffusing substance can enter into a limited body, which is a certain contradiction. Similarly, while maintaining a much larger flux through the “bottom” layer boundary we also come to certain collisions. In this regard, we propose to set the value of mass flux on one body surface, the value of the substance concentration on another and further determine the corresponding condition for the flux.

In the present work we investigate stochastic flows of an admixture substance in two-phase randomly nonhomogeneous stratified bodies under uniform

distribution of phases for the cases of both zero and non-zero constant initial admixture concentration on the basis of constructing solutions of initial-boundary value problems with stochastical coefficients in the form of Neumann series, which is useful for the procedure of averaging over the ensemble of phase configurations. Herewith series expansion is carried out in the neighborhood of a solution of the corresponding initial-boundary value problem for a homogeneous body.

## 2. Mathematical model of diffusion fluxes of admixture particles in stratified bodies

### 2.1. Diffusion equation for the function of mass flux

Let the process of admixture diffusion occur in a medium. In the general case the equation of mass balance has the form [13]

$$\frac{\partial c(\vec{r}, t)}{\partial t} = -\vec{\nabla} \cdot \vec{J}(\vec{r}, t) \tag{1}$$

where  $c(\vec{r}, t)$  is the concentration of admixture particles,  $\vec{J}(\vec{r}, t)$  is the mass flux of the diffusing substance,  $\vec{\nabla}$  is the Hamilton nabla-operator,  $\vec{r}$  is a radius-vector of the running point,  $t$  is time, the operation of scalar multiplication is marked by a point.

Let us act left on Equation (1) by the operator  $(-\vec{\nabla})$ :

$$\frac{\partial}{\partial t} (-\vec{\nabla} c(\vec{r}, t)) = \vec{\nabla} \otimes \vec{\nabla} \cdot \vec{J}(\vec{r}, t) \tag{2}$$

Here “ $\otimes$ ” is the tensor multiplication and besides  $\vec{\nabla} \otimes \vec{\nabla} = \nabla_i \nabla_j \vec{i}^i \otimes \vec{i}^j$  ( $i, j = \overline{1, 3}$ ), where  $\nabla_i$  is the symbol of the partial derivative,  $\vec{i}^i$  is the base vector (in the case of the Cartesian coordinate system  $\nabla_1 = \partial/\partial x$ ,  $\nabla_2 = \partial/\partial y$ ,  $\nabla_3 = \partial/\partial z$ ;  $\vec{i}^1 = \vec{i}$ ,  $\vec{i}^2 = \vec{j}$ ,  $\vec{i}^3 = \vec{k}$ ).

Let us multiply Equation (2) on the diffusion coefficient  $D(\vec{r})$ , which is accepted as time-independent, however it can be a function of space coordinates. Then we have

$$\frac{\partial}{\partial t} (-D(\vec{r}) \vec{\nabla} c(\vec{r}, t)) = D(\vec{r}) \vec{\nabla} \otimes \vec{\nabla} \cdot \vec{J}(\vec{r}, t) \tag{3}$$

Taking into account the relation between the mass flux and the particle concentration (the first Fick law) [14]

$$\vec{J}(\vec{r}, t) = -D(\vec{r}) \vec{\nabla} c(\vec{r}, t) \tag{4}$$

Equation (3) is written as

$$\frac{\partial \vec{J}(\vec{r}, t)}{\partial t} = D(\vec{r}) \vec{\nabla} \otimes \vec{\nabla} \cdot \vec{J}(\vec{r}, t) \tag{5}$$

Thus, we obtain the admixture diffusion equation represented by mass fluxes. In particular, in a one-dimensional case Equation (5) is reduced to the following:

$$\frac{\partial J(z, t)}{\partial t} = D(z) \frac{\partial^2 J(z, t)}{\partial z^2} \tag{6}$$

Note that since we have acted on the differential equation by an operator, then its solution can be determined up to an arbitrary function  $f$ , which satisfies

the condition  $\vec{\nabla} f = 0$ . Remark also that Equation (5) is valid for bodies with both deterministic and randomly nonhomogeneous multiphase structures. In particular, Equation (6) in the one-dimensional case describes the flow function in a stochastically nonhomogeneous stratified medium.

Moreover, not all formally admissible mathematical statements of initial-boundary value problems for Equations (5) and (6) are admissible from a physical point of view. Hence, we have to consider “mixed” statements of initial-boundary value problems which are presented below.

## ***2.2. Initial and boundary conditions of the first kind in problems of mass flux for a layer***

Consider the process of admixture substance diffusion in a layer of thickness  $z_0$  that contains sublayers, the location of which in the area of the body, generally speaking, is unknown.

Accept that the initial and boundary conditions of the first kind are satisfied for the flow function  $J(z, t)$ . In the initial moment of time there is no diffusion flow in the body. The admixture flow on the “upper” surface of the layer  $z = 0$  is constant, and particle concentration equals zero on the “lower” boundary of the strip  $z = z_0$ , namely

$$J(z, t)|_{t=0} = 0 \quad (7)$$

$$J(z, t)|_{z=0} = J_* \equiv \text{const}, \quad c(z, t)|_{z=z_0} = 0 \quad (8)$$

In this case the diffusion flow on the “lower” boundary is a function of time  $F(t)$  and we need to define additionally

$$J(z, t)|_{z=z_0} = F(t) \quad (9)$$

We shall determine the diffusion flow on the boundary  $z = z_0$ , *i.e.* the function  $F(t)$ , from the corresponding initial-boundary value problem for the migrating substance concentration.

Let us set the initial condition for the admixture particle concentration, equivalent to the initial condition for the flow of the substance (7), using a chemical potential, which is a continuous function on the inner boundaries of contact.

The first Fick law (4) with the use of the chemical potential  $\mu(z, t)$  is [15]

$$J(z, t) = -\tilde{L}(z) \frac{\partial \mu(z, t)}{\partial z} \quad (10)$$

where  $\tilde{L}(z)$  is the kinetic coefficient of mass transfer.

Taking into account relation (10) we represent the condition on the flow (7) as

$$\left. \frac{\partial \mu(z, t)}{\partial z} \right|_{t=0} = 0 \quad (11)$$

whence we obtain

$$\mu(z, t)|_{t=0} = \mu_* \equiv \text{const} \quad (12)$$

It follows from sufficient physical generals that the relation between the chemical potential and concentration is of a logarithmic nature [15, 16]

$$\mu(z, t) = \mu^0 + A \ln \gamma(z) c(z, t) \tag{13}$$

where  $\mu^0$  is the chemical potential for pure substance in the state determined by the values of absolute temperature  $T$  and pressure  $P$ ;  $A = RT/M$  is the coefficient, where  $R$  is the absolute gas constant,  $M$  is the atomic weight of admixture particles;  $\gamma(z)$  is the activity coefficient which can be presented for a two-phase body as

$$\gamma(z) = \begin{cases} \gamma_0 & z \in \Omega_0 \\ \gamma_1 & z \in \Omega_1 \end{cases} \tag{14}$$

here  $\Omega_j$  is the domain of phase  $j$  ( $j = 0; 1$ ),  $\sum_j \Omega_j = \Omega$ , where  $\Omega$  is the domain of the whole body.

If we linearize the relation (13) then we obtain the linear dependence of the chemical potential on the concentration in the form

$$\mu(z, t) = \mu^0 - A(1 - \gamma(z)c(z, t)) \tag{15}$$

The limits of satisfiability of the relation (15) are determined from comparison with experimental data.

From (15) we find the expression for the admixture concentration function

$$c(z, t) = \frac{1}{\gamma(z)} \left[ 1 + \frac{1}{A} (\mu(z, t) - \mu^0) \right] \tag{16}$$

Then, in the initial time moment for the concentration we obtain

$$c(z, t)|_{t=0} = \frac{1}{\gamma(z)} \left[ 1 + \frac{1}{A} (\mu(z, t) - \mu^0)|_{t=0} \right] \tag{17}$$

In particular, accepting the condition (12) we have

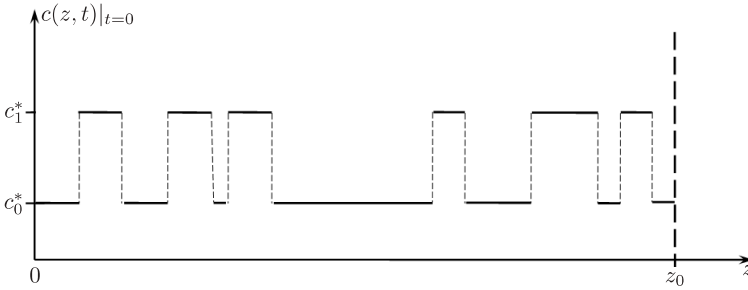
$$c(z, t)|_{t=0} = \frac{1}{\gamma(z)} \left[ 1 + \frac{1}{A} (\mu_* - \mu^0) \right] \tag{18}$$

Taking into account the representation (14), we write the condition (18) in the form

$$c(z, t)|_{t=0} = \begin{cases} [1 + (\mu_* - \mu^0)/A] / \gamma_0 & z \in \Omega_0 \\ [1 + (\mu_* - \mu^0)/A] / \gamma_1 & z \in \Omega_1 \end{cases} \tag{19}$$

Denote  $c_j^* = [1 + (\mu_* - \mu^0)/A] / \gamma_j$  ( $j = \overline{0, 1}$ ). Then,  $c(z, t)|_{t=0} = \{c_j^* \equiv \text{const}, z \in \Omega_j\}$ . Thus, we have obtained a piecewise-constant function of the initial concentration, a schematic drawing of which is shown in Figure 1. In this graph the y-axis represents the function  $c(z, t)|_{t=0}$ , the x-axis represents the spatial coordinate  $z$ . Note that there are jumps discontinuities of admixture concentration in the initial time on the boundaries of contact of the domains  $\Omega_j$  (Figure 1).

Note that the quantity  $c(z, t)|_{t=0}$  can be both random and deterministic depending on stochasticity or determinacy of the domain  $\Omega$ . Henceforth, suppose that the disposition of the domain  $\Omega_j$  is unknown, *i. e.* the coordinates of sublayer locations are random.



**Figure 1.** Schematic distribution of initial concentration in a two-phase stratified body

If the activity coefficients are close in different phases, *i.e.* we can assume that  $\gamma_0 \approx \gamma_1 \equiv \gamma_*$ , then  $c_0^* \approx c_1^* \equiv c_*$  and the condition (19) is as follows

$$c(z, t)|_{t=0} = c_* \equiv \text{const} \tag{20}$$

Henceforth, we shall consider the initial condition on the function of admixture concentration in the form (20), and singling out the separate case of absence of the admixture substance in the initial moment of time in a body

$$c(z, t)|_{t=0} = 0 \tag{21}$$

Remark that in the case of diffusion in stochastically nonhomogeneous bodies the initial condition (19) is random, and to construct solutions of initial-boundary value problems, in which the coefficients of differential equations and initial and boundary conditions are stochastic, it is required to develop an individual theory of mathematical physics using the theory of random fields.

### 3. Mathematical modeling diffusion flows of admixture in a randomly nonhomogeneous multilayered strip with uniform distribution of phases

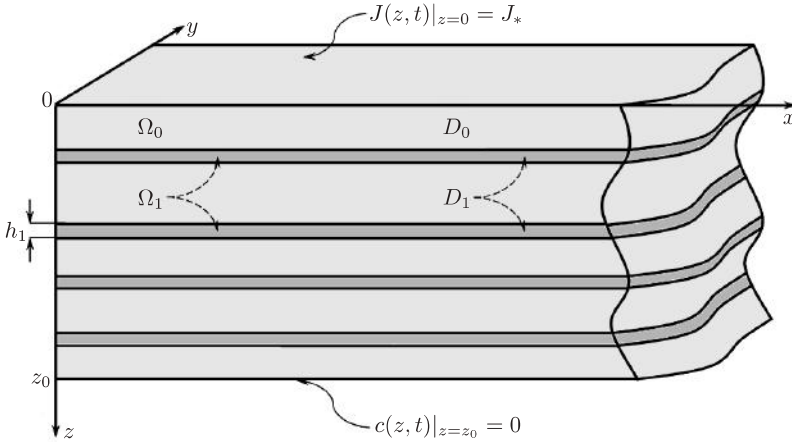
#### 3.1. Subject of inquiry and statement of the problem

Consider the admixture diffusion in a two-phase multilayered strip consisting of  $n_0$  sublayers of the phase  $j = 0$  (matrix) and  $n_1$  sublayers of the phase  $j = 1$  (inclusions). Here the coordinates of location of inclusions and the matrix, respectively, are unknown. Accept that phases in the body are disposed by the uniform law of distribution. We regard that the volume fraction of the matrix  $v_0$  is much larger than the volume fraction of inclusions, *i.e.*  $v_0 \gg v_1$ , and the coefficients of admixture diffusion are constant within the scope of each phase. One of the possible realizations of a two-phase multilayered structure is shown in Figure 2.

The diffusion coefficients in Equation (6) that is a random function of the spatial coordinate can be presented as

$$D(z) = \begin{cases} D_0 & z \in \Omega_0 \\ D_1 & z \in \Omega_1 \end{cases} \tag{22}$$

where  $\Omega_j = \bigcup_{i=1}^{n_j} \Omega_{ij}$  ( $j = \overline{0, 1}$ ,  $i = \overline{1, n_j}$ ),  $\Omega_{ij}$  is the  $i$ -th simply connected domain of the kind  $j$ .



**Figure 2.** Possible realization of a two-phase multilayered strip with uniform distribution of inclusions

The random flow of admixture particles  $J(z, t)$  in a multilayered strip is described by Equation (6). The constant diffusion flow  $J_*$  is supported on the body surface  $z = 0$ . And we assume that the concentration equals zero on the boundary  $z = z_0$  (Figure 2), *i.e.* the boundary conditions (8) are met. Let us consider the zero initial condition on the admixture flow (7) under zero (21) and nonzero constant (20) initial concentrations.

To construct a solution of the initial-boundary value problem (6)–(8) with the random diffusion coefficient we present it in terms of the random “function of structure” [17]  $\eta_{ij}(z) = \begin{cases} 1 & z \in \Omega_{ij} \\ 0 & z \notin \Omega_{ij} \end{cases}$ , where  $i = \overline{1, n_j}, j = 0; 1$ . Then, the admixture diffusion coefficient  $D(z)$  takes the form

$$D(z) = \sum_{i=1}^{n_0} D_0 \eta_{i0}(z) + \sum_{i=1}^{n_1} D_1 \eta_{i1}(z) \tag{23}$$

Herewith  $\sum_{i=1}^{n_0} \eta_{i0}(z) + \sum_{i=1}^{n_1} \eta_{i1}(z) = 1$  (the condition of body continuity). Note that the choice of a number of simply connected domains of both the matrix and the inclusion  $i$  is conventional.

Substituting such representation of coefficient  $D(z)$  in Equation (6) we obtain

$$\frac{\partial J(z, t)}{\partial t} - \left( \sum_{i=1}^{n_0} D_0 \eta_{i0}(z) + \sum_{i=1}^{n_1} D_1 \eta_{i1}(z) \right) \frac{\partial^2 J(z, t)}{\partial z^2} = 0 \tag{24}$$

In Equation (24) add and deduct the deterministic operator  $L_0(z, t)$

$$L_0(z, t) \equiv \frac{\partial}{\partial t} - D_0 \frac{\partial^2}{\partial z^2} \tag{25}$$

Then denoting the operators of Equation (24)

$$L(z, t) \equiv \frac{\partial}{\partial t} - \left( \sum_{i=1}^{n_0} D_0 \eta_{i0}(z) + \sum_{i=1}^{n_1} D_1 \eta_{i1}(z) \right) \frac{\partial^2}{\partial z^2}, \quad L_0(z, t) \equiv \frac{\partial}{\partial t} - D_0 \frac{\partial^2}{\partial z^2} \tag{26}$$

and accounting for the body continuity condition we obtain

$$L_0(z, t)J(z, t) = L_s(z, t)J(z, t) \tag{27}$$

where  $L_s(z, t) \equiv L_s(z) = (D_1 - D_0) \sum_{i=1}^{n_1} \eta_{i1}(z) \partial^2 / \partial z^2$ .

We shall find the solution of the initial-boundary value problem (27), (7), (8) in the form of the Neumann series.

**3.2. The integro-differential equation equivalent to the original initial-boundary value problem**

Considering the nonhomogeneity of a body structure as inner sources the solution of the initial-boundary value problem (27), (7), (8) can be presented by the sum of the solution of the homogeneous problem and the convolution of the Green function with the source [18]

$$J(z, t) = J_0(z, t) + \int_0^t \int_0^{z_0} G(z, z', t, t') L_s(z') J(z', t') dz' dt' \tag{28}$$

where  $J_0(z, t)$  is the solution of the homogeneous initial-boundary value problem,  $G(z, z', t, t')$  is the Green function of the problem (27), (7), (8), a deterministic function.

To find the solution of the homogeneous initial-boundary value problem

$$\frac{\partial J_0(z, t)}{\partial t} = D_0 \frac{\partial^2 J_0(z, t)}{\partial z^2} \tag{29}$$

$$J_0(z, t)|_{t=0} = 0; \quad J_0(z, t)|_{z=0} = J_* \equiv \text{const}, \quad J_0(z, t)|_{z=z_0} = F(t) \tag{30}$$

we must first determine the boundary condition for the flow function on the boundary  $z = z_0$ . For this purpose, we solve the initial-boundary value problem formulated for the function of migrating particle concentration  $c(z, t)$ . If the distribution of the concentration equals zero in the initial moment of time, the initial-boundary problem is as follows

$$\frac{\partial c(z, t)}{\partial t} = D_0 \frac{\partial^2 c(z, t)}{\partial z^2} \tag{31}$$

$$\partial c(z, t) / \partial z|_{z=0} = -J_* / D_0 \equiv \text{const}, \quad c(z, t)|_{z=z_0} = 0 \tag{32}$$

$$c(z, t)|_{t=0} = 0 \tag{33}$$

If the constant nonzero distribution of concentration in the strip is known in the initial moment, then the condition (33) takes the form

$$c(z, t)|_{t=0} = c_* \equiv \text{const} \tag{34}$$

The solution of the problem with zero initial concentration is obtained in the form

$$c(z, t) = \frac{J_*}{D_0} \left( z_0 - z - \frac{2}{z_0} \sum_{n=1}^{\infty} \frac{e^{-D_0 \xi_n^2 t} \cos(\xi_n z)}{\xi_n^2} \right) \tag{35}$$

And for the nonzero constant initial concentration it is found as follows

$$c(z, t) = \frac{J_*}{D_0} (z_0 - z) - \frac{2}{z_0} \sum_{n=1}^{\infty} e^{-D_0 \xi_n^2 t} \left( \frac{J_*}{D_0 \xi_n^2} + \frac{c_* (-1)^n}{\xi_n} \right) \cos(\xi_n z) \tag{36}$$

where  $\xi_n = \pi(2n - 1) / 2z_0$ .



Taking into account the relation between the mass flow and the particle concentration (4), from the solution (35) of the initial-boundary value problem (31)–(33) we obtain the expression in the homogeneous layer

$$J_0(z, t) = J_* \left( 1 - \frac{2}{z_0} \sum_{n=1}^{\infty} \frac{1}{\xi_n} e^{-D_0 \xi_n^2 t} \sin(\xi_n z) \right) \tag{37}$$

In particular, we have  $z = z_0$  on the boundary

$$J_0(z, t)|_{z=z_0} \equiv F(t) = J_* \left( 1 - \frac{2}{z_0} \sum_{n=1}^{\infty} \xi_n^{-1} (-1)^{n+1} e^{-D_0 \xi_n^2 t} \right) \tag{38}$$

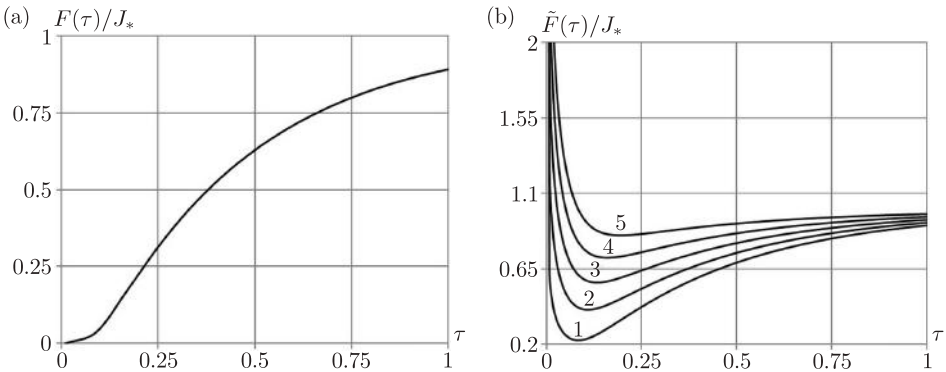
Similarly, for the initial-boundary value problem (31), (32), (34) we find

$$J_0(z, t) = J_* - \frac{2}{z_0} \sum_{n=1}^{\infty} e^{-D_0 \xi_n^2 t} \left( \frac{J_*}{\xi_n} + D_0 c_* (-1)^n \right) \sin(\xi_n z) \tag{39}$$

And for  $z = z_0$  we obtain the following boundary condition for the function of diffusion flow

$$J_0(z, t)|_{z=z_0} \equiv \tilde{F}(t) = J_* - \frac{2}{z_0} \sum_{n=1}^{\infty} (-1)^{n+1} e^{-D_0 \xi_n^2 t} \left( \frac{J_*}{\xi_n} + D_0 c_* (-1)^n \right) \tag{40}$$

In Figure 3 the behavior of functions  $F(t)$  at zero (a) and  $\tilde{F}(t)$  at nonzero (b) concentrations in the initial moment under different values of  $c_*/J_* = 0.1; 0.2; 0.3; 0.4; 0.5$  (curves 1–5 in Figure 3b) for the dimensionless time  $\tau = D_0 t/z_0^2$  is shown. It should be remarked that in the case of the zero initial concentration,  $F(t)$  is a steadily increasing function. However, at certain constant nonzero initial concentration, the function  $\tilde{F}(t)$  slumps and after reaching a local minimum begins to grow. In addition, the higher the value of ratio  $c_*/J_*$ , the faster the function  $F(t)$  goes on the steady-state regime.

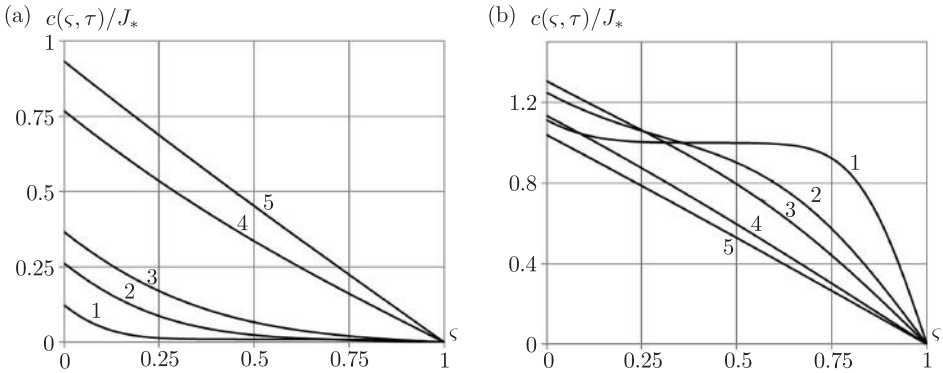


**Figure 3.** Functions  $F(\tau)/J_*$  for zero (a) and  $\tilde{F}(\tau)/J_*$  for nonzero initial conditions at different values of ratio  $c_*/J_*$  (b)

Numerical calculations are performed in the dimensionless variables [19]

$$\zeta = z/z_0, \quad \tau = D_0 t/z_0^2 \tag{41}$$

Figure 4 illustrates the peculiar distributions of admixture concentrations at nonzero constant initial concentration in the homogeneous layer according to the formula (36) in different time moments  $\tau = 0.01; 0.05; 0.1; 0.5; 1$  (curves 1–5) under  $c_*/J_* = 0.01$  (a) and  $c_*/J_* = 1$  (b). The condition  $c_*/J_* = 1$  means that the case under consideration is maintenance of small admixture flows on the layer boundary  $J_*$  so long as it follows  $0 \leq J_* \leq 1$  from the definition of concentration; moreover we have  $J_* \ll 1$  from the definition of admixture concentration. Figure 5 shows the dependence of mass flows in the homogeneous strip on the value of ratio  $c_*/J_* = 0.1; 0.2; 0.3; 0.4; 0.5$  (curves 1–5) in time moments  $\tau = 0.01$  (a) and  $\tau = 0.1$  (b).

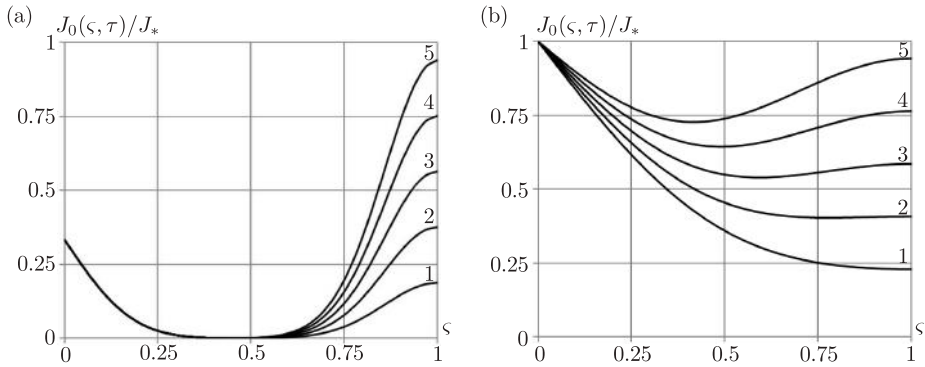


**Figure 4.** Distributions of concentration in a layer at nonzero initial condition in different moments for  $c_*/J_* = 0.01$  (a) and  $c_*/J_* = 1$  (b)

With increasing time for the diffusion process at the ratio  $c_*/J_* = 0.01$  the admixture concentration increases in the body (Figure 4a) till it gets the steady-state regime (curve 5 in Figure 4a). In the case  $c_*/J_* = 1$  for small times, in particular  $\tau = 0.01$ , the area of approximately constant concentration  $\zeta \in [0.15; 0.65]$  persists (curve 1 in Figure 4b). Moreover in the near-surface regions ( $\zeta = 0$ ), where the mass source acts, an increase in the admixture concentration is observed with increasing time in the interval  $\tau \in (0; 0.1]$  (curves 1–3 in Figure 4b), with a further increase in time the concentration decreases (curves 4, 5 in Figure 4b). Raising the ratio  $c_*/J_*$  leads to an increase in the admixture concentration as well as the diffusion flow in the homogeneous strip (Figure 5), but for small times, namely for  $\tau = 0.01$ , formation of a maximum (global) of the diffusion flow is observed near the surface  $\zeta = 1$  (Figure 5a).

The Green function is a solution of the initial-boundary value problem of diffusion from a point source under zero initial and boundary conditions, *i.e.* it is the solution of the following initial-boundary value problem

$$\frac{\partial G(z, z', t, t')}{\partial t} - D_0 \frac{\partial^2 G(z, z', t, t')}{\partial z^2} = \delta(t - t') \delta(z - z') \tag{42}$$



**Figure 5.** Dependence of mass flows in a layer on the value of ratio  $c_*/J_*$  in the moments  $\tau = 0.01$  (a) and  $\tau = 0.1$  (b)

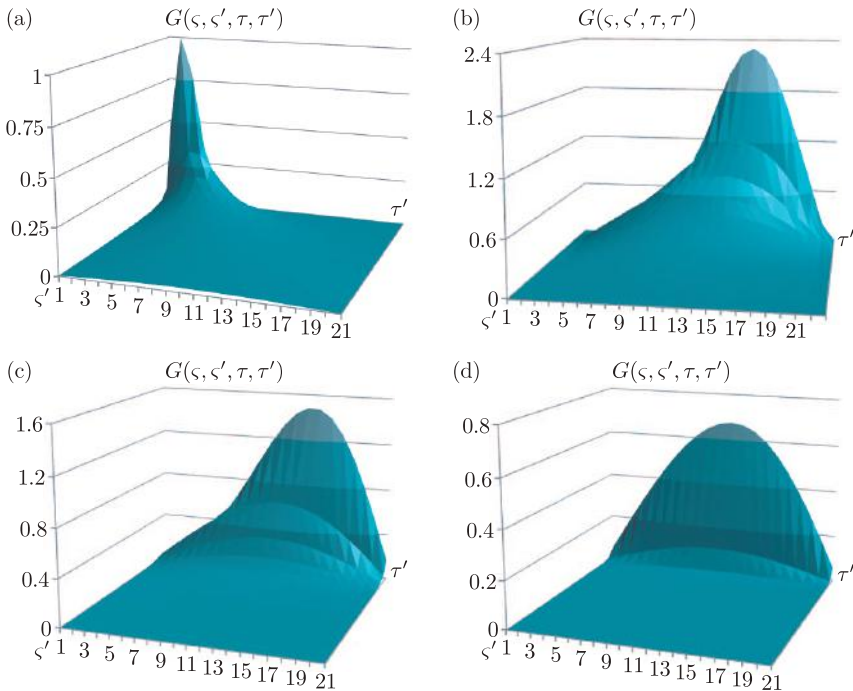
$$G(z, z', t, t')|_{t=0} = 0, \quad G(z, z', t, t')|_{z=0} = 0, \quad G(z, z', t, t')|_{z=z_0} = 0 \quad (43)$$

We obtain it in the form

$$G(z, z', t, t') = \frac{\theta(t-t')}{z_0} \sum_{k=1}^{\infty} e^{-D_0 y_k^2 (t-t')} [\cos(y_k(z-z')) - \cos(y_k(z+z'))] \quad (44)$$

where  $\theta(t-t')$  is the unit step Heaviside function,  $y_k = k\pi/z_0$ .

Examples of surfaces formed by the Green function, are built in the dimensionless variables (41) and shown in Figure 6 in points  $(\varsigma, \tau) = (0.0125; 0.125)$  (a),



**Figure 6.** The Green function in points  $(\varsigma, \tau) = (0.0125; 0.125)$  (a),  $(\varsigma, \tau) = (0.6875; 0.625)$  (b),  $(\varsigma, \tau) = (0.6875; 1.375)$  (c),  $(\varsigma, \tau) = (0.6875; 3.875)$  (d)

$(\varsigma, \tau) = (0.6875; 0.625)$  (b),  $(\varsigma, \tau) = (0.6875; 1.375)$  (c) and  $(\varsigma, \tau) = (0.6875; 3.875)$  (d). The spatial coordinate  $\varsigma'$  is set along the abscise axis and the time variable  $\tau'$  is set along the axis of ordinates.

Thus, the original initial-boundary value problem has been reduced to the equivalent integro-differential equation (28) with random kernel. This equation is the Volterra equation of the second kind in time and the Hammerstein equation in the spatial variable [20] and contains the diffusion flow in a homogeneous layer (37) or (39) and the deterministic Green function (44).

### 3.3. Neumann series

The solution of the integro-differential equation (28) with random kernel is found by the method of successive iterations. We accept the solution of the homogeneous problem as an initial approximation  $J^{(0)}(z, t) = J_0(z, t)$ . Then, we obtain the following recurrence relations

$$\begin{aligned}
 J^{(1)}(z, t) &= J_0(z, t) + \int_0^t \int_0^{z_0} G(z, z', t, t') L_s(z') J^{(0)}(z', t') dz' dt' \\
 J^{(2)}(z, t) &= J_0(z, t) + \int_0^t \int_0^{z_0} G(z, z', t, t') L_s(z') J^{(1)}(z', t') dz' dt' \\
 &\dots \\
 J^{(n)}(z, t) &= J_0(z, t) + \int_0^t \int_0^{z_0} G(z, z', t, t') L_s(z') J^{(n-1)}(z', t') dz' dt' \\
 &\dots
 \end{aligned}
 \tag{45}$$

The general term of the constructed sequence of functions  $J^{(0)}, J^{(1)}, \dots, J^{(n)}, \dots$  can be represented as

$$\begin{aligned}
 J^{(n)}(z, t) &= J_0(z, t) + \int_0^t \int_0^{z_0} G(z, z', t, t') L_s(z') J_0(z', t') dz' dt' + \dots + \\
 &+ \int_0^t \int_0^{z_0} G(z, z', t, t') L_s(z') \left[ \int_0^{t'} \int_0^{z_0} G(z', z'', t', t'') L_s(z'') \times \dots \times \right. \\
 &\times \int_0^{t^{(n-2)}} \int_0^{z_0} G(z^{(n-2)}, z^{(n-1)}, t^{(n-2)}, t^{(n-1)}) L_s(z^{(n-1)}) \times \\
 &\left. \times J_0(z^{(n-1)}, t^{(n-1)}) dz^{(n-1)} dt^{(n-1)} \dots \right] dz' dt' + R_n(z, t)
 \end{aligned}
 \tag{46}$$

where  $R_n(z, t)$  is difference between the  $n$ -th and  $(n - 1)$ -th terms of the sequence, namely

$$\begin{aligned}
 R_n(z, t) &= \int_0^t \int_0^{z_0} G(z, z', t, t') L_s(z') \int_0^{t'} \int_0^{z_0} G(z', z'', t', t'') L_s(z'') \dots \times \\
 &\times \int_0^{t^{(n-1)}} \int_0^{z_0} G(z^{(n-1)}, z^{(n)}, t^{(n-1)}, t^{(n)}) L_s(z^{(n)}) J_0(z^{(n)}, t^{(n)}) dz^{(n)} dt^{(n)} \dots dz' dt'
 \end{aligned}
 \tag{47}$$

Assign such series

$$J(z, t) \equiv J_0(z, t) + \sum_{n=1}^{\infty} R_n(z, t)
 \tag{48}$$

to the constructed sequence of functions. This series is an integral Neumann series.

Remark that, so far as the function  $J_0(z, t)$  is continuously differentiable then acting on it by the operator  $L_s(z)$  we obtain the expression

$$L_s(z) J_0(z, t) = (D_1 - D_0) \sum_{i=1}^{n_1} \eta_{i1}(z) \frac{\partial^2 J_0(z, t)}{\partial z^2}
 \tag{49}$$

**Statement.** If the diffusion coefficients  $D_0, D_1$  are bounded and  $D_0 \neq 0$ , then such conditions are satisfied for the Green function  $G(z, z', t, t')$  and the diffusion flow in the homogeneous body  $J_0(z, t)$

- (1)  $|G(z, z', t, t')| \leq K_1 < \infty, \quad \forall z, z' \in [0, z_0], \quad \forall t' \in [0, t], \quad \forall t \in [0, \bar{\tau}] \quad (\bar{\tau} < \infty)$
  - (2)  $|L_s(z) G(z, z', t, t')| \leq K_2 < \infty, \quad \forall z, z' \in [0, z_0], \quad \forall t' \in [0, t], \quad \forall t \in [0, \bar{\tau}]$
  - (3)  $|L_s(z) J_0(z, t)| \leq K_3 < \infty, \quad \forall z \in [0, z_0], \quad \forall t \in [0, \bar{\tau}]$
- (50)

Remark that as the condition of zero initial concentration is the partial case of the nonzero constant initial concentration condition, then the Statement is true in the absence of an admixture substance in the body in the initial moment, too.

**Theorem 1.** When the conditions of the Statement are satisfied, the Neumann series (48) is absolutely and uniformly convergent.

**Proof.** Taking into account the relations (50) we obtain the following estimate for the general term of the Neumann series

$$|R_n| \leq K_1 K_2^{n-1} K_3 \frac{(z_0 t)^n}{n!}
 \tag{51}$$

Whereas a majorant series with the positive general term  $K_1 K_2^{n-1} K_3 (z_0 t)^n / n!$  converges at  $n \rightarrow \infty$  for arbitrary values  $K_1, K_2, K_3, z_0, t$ , then the sequence of partial sums of the series (48)  $\{J^{(n)}(z, t)\}$  is absolutely and uniformly convergent at  $n \rightarrow \infty$  after the Weierstrass criterion, namely

$$\lim_{n \rightarrow \infty} J^{(n)}(z, t) = J(z, t)
 \tag{52}$$

Theorem 1 is proved.

**Theorem 2.** The function  $J(z, t) \equiv J_0(z, t) + \sum_{n=1}^{\infty} R_n(z, t)$  is a solution of the integro-differential equation (28).

**Proof.** Substitute the series (48) into Equation (28), then we obtain

$$\begin{aligned}
 J_0(z, t) + \sum_{n=1}^{\infty} R_n(z, t) &= J_0(z, t) + \\
 + \int_0^t \int_0^{z_0} G(z, z', t, t') L_s(z') &\left[ J_0(z, t) + \sum_{n=1}^{\infty} R_n(z, t) \right] dz' dt'
 \end{aligned} \tag{53}$$

Use the definition of function  $R_n(z, t)$

$$\begin{aligned}
 &\int_0^t \int_0^{z_0} G(z, z', t, t') L_s(z') J_0(z, t) dz' dt' + \int_0^t \int_0^{z_0} G(z, z', t, t') L_s(z') \times \\
 &\times \int_0^{t'} \int_0^{z_0} G(z', z'', t', t'') L_s(z'') J_0(z'', t'') dz'' dt'' dz' dt' + \dots = \\
 &= \int_0^t \int_0^{z_0} G(z, z', t, t') L_s(z') J_0(z, t) dz' dt' + \\
 &+ \int_0^t \int_0^{z_0} G(z, z', t, t') L_s(z') \int_0^{t'} \int_0^{z_0} G(z', z'', t', t'') L_s(z'') J_0(z'', t'') dz'' dt'' dz' dt' + \dots
 \end{aligned} \tag{54}$$

The obtained identity proves Theorem 2.

Let us find an estimate for the sum of the remainders of the series (48)  $S_n = \sum_{k=n+1}^{\infty} R_k(z, t)$ . As long as the inequality (51) takes place, then performing summation of both right- and left-hand sides of this relation, we obtain

$$|S_n| \leq \frac{K_1 K_3}{K_2} \exp(K_2 z_0 \bar{\tau}) \left[ 1 - \frac{1}{n!} \Gamma(n+1, K_2 z_0 \bar{\tau}) \right] \tag{55}$$

where  $\Gamma(n+1, K_2 z_0 \bar{\tau}) = \int_{K_2 z_0 \bar{\tau}}^{\infty} x^n e^{-x} dx$  is the additional incomplete Gamma-function.

Remark that in previous investigations [21, 17] for convergence of the Neumann series the condition of boundedness of the of inclusion disposition domain is imposed. As follows from the Statement and Theorem 1 this condition is not necessary for a nonstationary case.

### 3.4. Averaging the diffusion flow over the ensemble of phase configurations

To find the averaged diffusion flow we restrict ourselves by two first terms of the Neumann series (48)

$$J(z, t) \approx J_0(z, t) + (D_1 - D_0) \int_0^t \int_0^{z_0} G(z, z', t, t') \sum_{i=1}^{n_1} \eta_{i1}(z') \frac{\partial J_0(z', t')}{\partial z^2} dz' dt' \tag{56}$$

Average the expression (56) over the ensemble of phase configurations with uniform distribution of sublayers in the body taking into account the fact that all sublayers of the inclusion phase have the same characteristic (average) thickness  $h_1$  (Figure 2, and the random coordinate characterizing the inclusion location is the coordinate of the “upper” boundary of the inclusion  $z_{i1}$  ( $i = \overline{1, n_1}$ ). Then using the relation (49) we have

$$\langle J_0(z, t) \rangle_{\text{conf}} = J_0(z, t), \quad \langle L_s(z') \rangle_{\text{conf}} = (D_1 - D_0) \sum_{i=1}^{n_1} \langle \eta_{i1}(z') \rangle_{\text{conf}} \frac{\partial^2}{\partial z^2} \tag{57}$$

Take into account that

$$\eta_{i1}(z') = \begin{cases} 1, & z' \in [z_{i1}; z_{i1} + h_1] \\ 0, & z' \notin [z_{i1}; z_{i1} + h_1] \end{cases} = \begin{cases} 1, & z' - z_{i1} \in [0; h_1] \\ 0, & z' - z_{i1} \notin [0; h_1] \end{cases} = \eta_{i1}(z' - z_{i1}) \tag{58}$$

Then

$$\sum_{i=1}^{n_1} \langle \eta_{i1}(z') \rangle = \sum_{i=1}^{n_1} \langle \eta_{i1}(z' - z_{i1}) \rangle = \sum_{i=1}^{n_1} \frac{1}{V} \int_0^{z_0 - h_1} \eta_{i1}(z' - z_{i1}) dz_{i1} = \sum_{i=1}^{n_1} \frac{1}{V} \int_0^{z'} \eta_{i1}(x) dx \tag{59}$$

where  $x = z' - z_{i1}$ . Therefore we have

$$\sum_{i=1}^{n_1} \frac{1}{V} \int_0^{z'} \eta_{i1}(x) dx = \frac{n_1}{V} \int_0^{z'} dx = \frac{z' n_1}{V} = \frac{z' n_1}{V} \cdot \frac{h_1}{h_1} = \frac{v_1 z'}{h_1} \tag{60}$$

for  $z' \leq h_1$  and

$$\sum_{i=1}^{n_1} \frac{1}{V} \int_0^{z'} \eta_{i1}(x) dx = \frac{n_1}{V} \int_0^{h_1} dx = \frac{h_1 n_1}{V} = v_1 \tag{61}$$

for  $z' \geq h_1$ . Then finally we obtain

$$\sum_{i=1}^{n_1} \langle \eta_{i1}(z') \rangle = \begin{cases} v_1 z' / h_1, & z' \leq h_1 \\ v_1, & z' \geq h_1 \end{cases} \tag{62}$$

Substituting the expressions (57) and (62) into (56) we obtain the formula for determination of the flow of admixture particles in a multilayered strip averaged over the ensemble of phase configurations with the uniform distribution function

$$\begin{aligned} \langle J(z, t) \rangle_{\text{conf}} = & J_0(z, t) + (D_1 - D_0) \int_0^t \left[ \frac{v_1}{h_1} \int_0^{h_1} z' G(z, z', t, t') \frac{\partial^2 J_0(z', t')}{\partial z^2} dz' + \right. \\ & \left. + v_1 \int_{h_1}^{z_0} G(z, z', t, t') \frac{\partial^2 J_0(z', t')}{\partial z^2} dz' \right] dt' \end{aligned} \tag{63}$$

If in the relation (63) we substitute the expressions for the Green function (44) and the diffusion flow in the homogeneous layer under zero initial concentration (37), then we obtain

$$\begin{aligned} \frac{1}{J_*} \langle J(z,t) \rangle_{\text{conf}} = & 1 - \frac{2}{z_0} \sum_{n=1}^{\infty} \frac{1}{\xi_n} e^{-D_0 \xi_n^2 t} \sin(\xi_n z) + \\ & + \frac{2v_1(D_1 - D_0)}{z_0^2 D_0} \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{\xi_n \bar{A}_{kn}}{y_k^2 - \xi_n^2} \left[ e^{-D_0 \xi_n^2 t} - e^{-D_0 y_k^2 t} \right] \sin(y_k z) \end{aligned} \tag{64}$$

In the case of nonzero constant initial concentration of the admixture we substitute the expression (39) into the formula (63) and we have

$$\begin{aligned} \frac{1}{J_*} \langle J(z,t) \rangle_{\text{conf}} = & 1 - \frac{2}{z_0} \sum_{n=1}^{\infty} e^{-D_0 \xi_n^2 t} \left( \frac{1}{\xi_n} + (-1)^n D_0 \frac{c_*}{J_*} \right) \sin(\xi_n z) + \\ & + \frac{2v_1(D_1 - D_0)}{z_0^2 D_0} \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{\xi_n \bar{A}_{kn}}{y_k^2 - \xi_n^2} \left( 1 + (-1)^n D_0 \frac{c_*}{J_*} \xi_n \right) \left( e^{-D_0 \xi_n^2 t} - e^{-D_0 y_k^2 t} \right) \sin(y_k z) \end{aligned} \tag{65}$$

Here  $\bar{A}_{kn} = \frac{\cos[(y_k - \xi_n)h_1]}{h_1(y_k - \xi_n)^2} - \frac{\cos[(y_k + \xi_n)h_1]}{h_1(y_k + \xi_n)^2} - \frac{4y_k \xi_n}{h_1(y_k^2 - \xi_n^2)^2} + (-1)^{k+n} \frac{2y_k}{y_k^2 - \xi_n^2}$ .

Note that we have not considered the specific form of initial and boundary conditions in the formula (63), which makes it possible to apply it for different kinds of initial and boundary conditions.

### 3.5. Numerical analysis of averaged diffusion flows

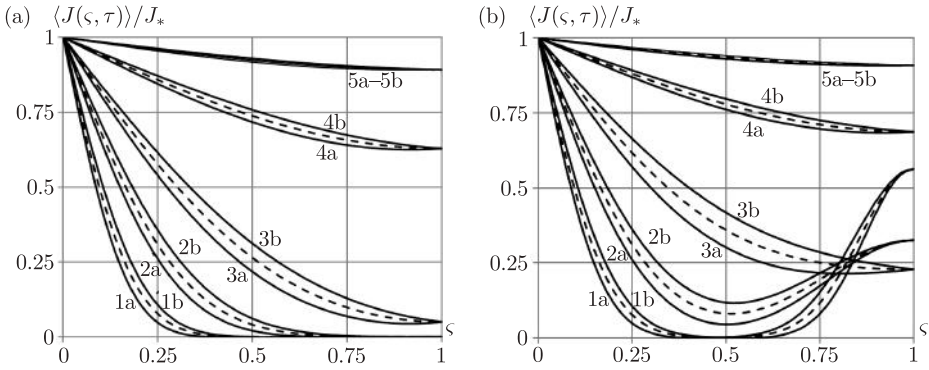
This section is devoted to the numerical analysis of averaged diffusion flows according to the found calculation formulae (64) and (65). These calculations were carried out in the dimensionless variables (41). We set the following basic parameters of the problem:  $\tau = 0.1$ ;  $v_1 = 0.2$ ;  $h_1 = 0.01$ ;  $c_*/J_* = 0.1$ .

Figure 7 shows the mass flow distributions in the strip at zero (a) and nonzero constant (b) initial conditions on the concentration. Curves 1–5 correspond to different moments of the dimensionless time  $\tau = 0.01$ ; 0.03; 0.1; 0.5; 1. Figure 8 illustrates the behavior of the averaged diffusion flow at zero (a) and nonzero (b) initial admixture concentrations for different values of the volume fraction of inclusions  $v_1 = 0.05$ ; 0.1; 0.2 (curves 1–3 respectively). In Figures 7 and 8 the curves ‘a’ are used for  $D_1/D_0 = 0.01$ , the curves ‘b’ for  $D_1/D_0 = 2$ . The dashed lines mark the corresponding flows in the homogeneous layer with matrix characteristics.

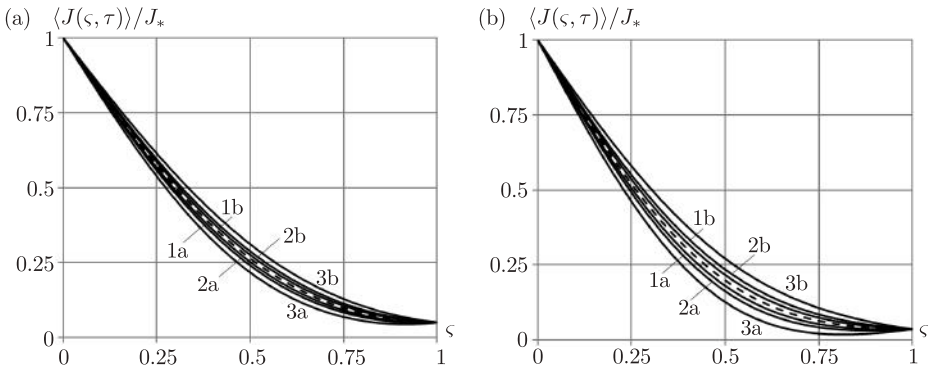
Figure 9a shows the distributions of averaged mass flows at zero initial concentration for different values of the ratio of diffusion coefficients  $D_1/D_0 = 0.01$ ; 0.5; 2; 5; 10; 15 (curves 1–6 respectively), and Figure 9b illustrates diffusion flows at nonzero constant initial admixture concentration for different values of the ratio  $c_*/J_* = 0.01$ ; 0.1; 0.2; 0.3; 0.4 (curves 1–5 respectively).

At the zero initial concentration of admixture particles the averaged flows are always steadily decreasing functions for  $D_1 < D_0$  (Figure 7a), which time-

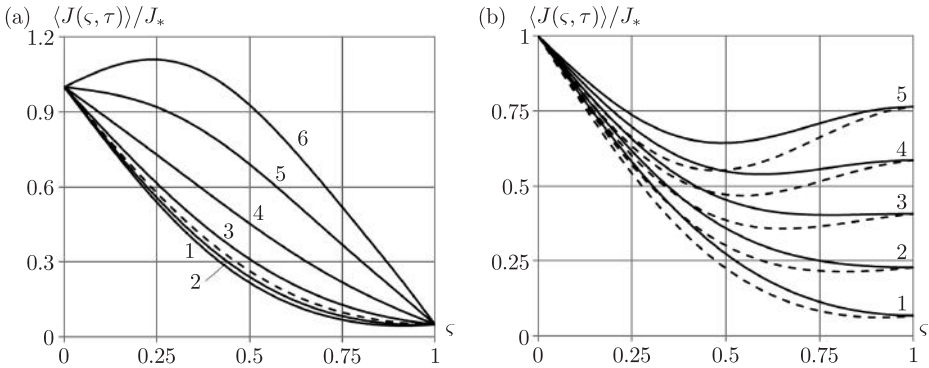




**Figure 7.** Distributions of mass flows in the strip in different moments of time at zero (a) and nonzero (b) initial concentrations



**Figure 8.** Distributions of mass flows in the strip depending on different values of volume fraction of inclusions at zero (a) and nonzero (b) initial concentrations



**Figure 9.** Distributions of mass flows in the strip at zero initial concentration for different values of  $D_1/D_0$  (a) and at nonzero initial concentration for different values of  $c_*/J_*$  (b)

increase in the whole body till they get a steady-state regime. In the case of nonzero constant initial concentration (Figure 7b) the behavior of the averaged diffusion flows for small times differs substantially from those with the zero initial concentration. With increasing time for the diffusion process at the ratio

$c_*/J_* = 0.01$  the admixture concentration increases in the body (Figure 4a) till it gets a steady state regime (curve 5 in Figure 4a). The diffusion flow decreases from the boundary  $\varsigma = 0$ , it is close to zero in the middle of the layer and increases rapidly near the boundary  $\varsigma = 1$  (curves 1a and 1b in Figure 7b), this explains the necessity to provide the condition of equality of zero admixture concentration on the boundary  $\varsigma = 1$ .

If the coefficient of admixture diffusion in the sublayers is less than in the matrix, the flow in a nonhomogeneous body is always smaller than in a homogeneous one. In an opposite case the flow in the multilayered strip is larger than in the homogeneous layer (Figure 7). In addition, flows in homogeneous and nonhomogeneous strips coincide on the layer boundary.

It should be remarked that with the growth of the volume fraction of inclusions the values of mass flow decrease for  $D_1 < D_0$  and increase for  $D_1 > D_0$  at both zero and nonzero initial concentration (Figure 8). In particular, increasing the volume fraction of inclusions from 0.1 to 0.2 leads to decreasing the averaged flow to 9% in the middle of the layer for  $D_1 < D_0$  and increasing to 9% for  $D_1 > D_0$ .

Raising the ratio of diffusion coefficients  $D_1/D_0$  at the zero initial concentration of admixture in the body leads to an increase in the averaged diffusion flow (Figure 9a), in addition to that, for large values of the ratio, the flow growth from the surface is observed, where the mass source acts (curve 6 in Figure 9a). The dependence of the function  $\langle J(\varsigma, \tau) \rangle / J_*$  at the nonzero initial concentration on the values of the ratio  $D_1/D_0$  is similar to the case of zero initial concentration.

The occurrence of admixture particles in the body in the initial moment significantly affects both the behavior and values of the admixture flow function. For small ratios  $c_*/J_*$  the admixture flow in both the homogeneous layer and the multilayered strip is a steadily decreasing function (curve 1a in Figure 9b). With the growth of initial concentration  $c_*$  the flow increases near the surface of the layer  $\varsigma = 1$  which can cause a local minimum in the middle of the body (curves 4a and 5a in Figure 9b). Changing the characteristic thickness of sublayers at the same volume fraction of inclusions, *i.e.* a change of the number of inclusions, has little effect on the values of the averaged diffusion flow for both zero and nonzero constant initial concentration (difference in the third significant digit).

## 4. Conclusion

Summing up, in this paper we have proposed a new approach to the mathematical description of admixture diffusion fluxes in bodies with a randomly non-homogeneous multiphase structure under which initial-boundary value problems are formulated directly for the function of mass flux. The diffusion equation for the flux of migration particles has been obtained on the basis of the equation of mass balance. We have justified the initial and boundary conditions for the flow in a layer for the avoidance of contradictions, for example, to prevent a case of an unlimited amount of substance in a limited body. The integro-differential equation

for the diffusion flow has been constructed that is equivalent to the original initial-boundary value problem. The solution of the equation has been found in the form of the integral Neumann series. We have formulated and proved a theorem on the absolute and uniform convergence of the Neumann series. Such a theorem has been proved for the first time for models of diffusion processes in stochastically nonhomogeneous bodies where a random structure is taken into account in the coefficients of the problem. The theorem of the existence of a solution for the corresponding integro-differential equation has been proved. We emphasize that we have proved the existence of a solution of the integro-differential equation and convergence of the corresponding integral series when the kernel of operator contains a random function of coordinates as a multiplier. With that it is not limited by any restrictions on the density of the distribution function of phases in the body. That is to say that the proposed methodology for study of such a class of diffusion problems is valid for arbitrary configuration of phases.

The stochastic admixture flow has been averaged over the ensemble of phase configurations in cases where the admixture is absent in the body in the initial moment or its constant nonzero initial distribution is known. Calculation formulae have been found for the averaged mass flow in a multilayered strip with a uniform distribution of phases. On this basis we have created software containing program modules for computation of admixture concentration and diffusion flow in the homogeneous body at zero and nonzero constant initial concentration of the migrating substance; The Green function; the diffusion flow averaged over the ensemble of phase configurations in three- and multilayered strip where phases distribute by the uniform law at zero and nonzero constant initial concentration of admixture particles. Simulation of the averaged diffusion flows in two-phase stratified bodies has been carried out. We have established that the ratio of the admixture diffusion coefficients in the inclusions and in the matrix affect most the behavior and values of the averaged mass flow, while the diffusion flow almost does not depend on the characteristic thickness of sublayers. It is shown that independently from the kind of the inclusion, the distribution of the diffusion flow in the nonhomogeneous body is always smaller than the flow in the body without sublayers, if the admixture diffusion coefficient in the inclusions is greater than the coefficient in the basic phase, and vice versa.

The designed software was applied to estimate on-stream time and efficiency of filters of cleaning urban municipal wastewaters, sewage from recreation on the beach and sewage brought by cesspoolage trucks from unducted territories. Here the results are used in the technique segment for cleaning urban runoff in biological reactors. In particular, an algorithm is developed and established for the relation between the costs of production of these arrangements and the level of their modern processibility in provision of the set indicators and parameters of water cleaning.

Finally, in the present paper we have considered the case of uniform distribution of layered inclusions only. In the next paper we shall propose a study

of averaging flows of admixture in two-phase stratified bodies under non-uniform distributions of inclusions, where the region of the most probable disposition of inclusions is located near the layer surface on which a mass source acts, in the vicinity of another body boundary and in the middle of the strip.

### Appendix. Proof of statement on function boundedness (50)

1. First show satisfaction of the inequality (1) in the formulae (50). The general term of the series (44) in the domain  $[0, z_0] \cup [[0, \bar{\tau}] \cap \{t = t'\}]$  can be estimated as

$$\left| e^{-D_0 y_k^2 (t-t')} \sin y_k z' \sin y_k z \right| \leq e^{-D_0 y_k^2 (t-t')} \tag{66}$$

The series  $\sum_{k=0}^{\infty} e^{-D_0 y_k^2 (t-t')}$  is absolutely convergent by the D'Alembert criterion. Then, by the Weierstrass criterion, the series  $\sum_{k=0}^{\infty} e^{-D_0 y_k^2 (t-t')} \sin y_k z'$   $\sin y_k z$  is absolutely and uniformly convergent, and therefore the sequence of its partial sums is also absolutely and uniformly convergent. Moreover, the series is bounded so long as a convergent sequence in the metric space is bounded. That is, the function  $g(z, z', t, t') = \sum_{k=1}^{\infty} e^{-D_0 y_k^2 (t-t')} [\cos(y_k(z - z')) - \cos(y_k(z + z'))]$  is bounded for  $\forall z, z' \in [0, z_0], \forall t, t' \in [0, \bar{\tau}]$  except the point  $t = t'$ . Since  $\theta(t - t') \leq 1$  for  $\forall t, t'$  then the function  $G(z, z', t, t')$  is also bounded for  $\forall z, z' \in [0, z_0], \forall t, t' \in [0, \bar{\tau}]$  except for the point  $t = t'$ .

Let us show the boundedness of the function  $g(z, z', t, t')$  in the point  $t = t'$ . For this aim we use the property that a continuous function is bounded on a closed interval and show that the function  $g(z, z', t, t')$  in the domain  $[0, z_0] \cup [[0, \bar{\tau}] \cap \{t = t'\}]$  is a continuous function of its arguments.

From the definition of the function continuity in a point we have that for  $\forall \varepsilon > 0, \forall \delta > 0$  such that from the condition  $\forall t, |t - t'| < \delta$  it follows  $|g(z, z', t, t') - g(z, z', t', t')| < \varepsilon$ .

Use the known series [22]

$$\sum_{k=1}^{\infty} \frac{k}{k^2 + a^2} \sin kx = \frac{\pi \operatorname{sh}(\pi - x)a}{2 \operatorname{sh} \pi a} \tag{67}$$

Differentiating this expression with respect to the variable  $x$  we obtain

$$\sum_{k=1}^{\infty} \frac{k^2}{k^2 + a^2} \cos kx = -\frac{\pi a \operatorname{ch}(\pi - x)a}{2 \operatorname{sh} \pi a} \tag{68}$$

Taking into account (68) and following the inequality [23]

$$e^z < (1 - z)^{-1} \quad (z < 1) \tag{69}$$

we have

$$\begin{aligned}
 |g(z, z', t, t') - g(z, z', t', t')| &< \frac{1}{z_0} \left| \sum_{k=1}^{\infty} \cos y_k (z - z') \left[ \frac{1}{1 + D_0 y_k^2 \delta} - 1 \right] \right| + \\
 + \frac{1}{z_0} \left| \sum_{k=1}^{\infty} \cos y_k (z + z') \left[ \frac{1}{1 + D_0 y_k^2 \delta} - 1 \right] \right| &= \frac{1}{z_0} \left| \sum_{k=1}^{\infty} \cos k \frac{\pi(z - z')}{z_0} \left[ -\frac{k^2}{k^2 + z_0^2 / D_0 \pi^2 \delta} \right] \right| + \\
 + \frac{1}{z_0} \left| \sum_{k=1}^{\infty} \cos k \frac{\pi(z + z')}{z_0} \left[ -\frac{k^2}{k^2 + z_0^2 / D_0 \pi^2 \delta} \right] \right| &= \frac{1}{z_0} \left| \frac{\operatorname{ch}(z_0 / \sqrt{D_0 \delta} - (z - z') / \sqrt{D_0 \delta})}{2\sqrt{D_0 \delta} \operatorname{sh}(z_0 / \sqrt{D_0 \delta})} \right| + \\
 + \frac{1}{z_0} \left| \frac{\operatorname{ch}(z_0 / \sqrt{D_0 \delta} - (z + z') / \sqrt{D_0 \delta})}{2\sqrt{D_0 \delta} \operatorname{sh}(z_0 / \sqrt{D_0 \delta})} \right| &\leq \frac{1}{\sqrt{D_0 \delta}} \operatorname{ch} \left( \frac{z_0}{\sqrt{D_0 \delta}} \right) \operatorname{cth} \left( \frac{z_0}{\sqrt{D_0 \delta}} \right) = \varepsilon
 \end{aligned} \tag{70}$$

That is, for arbitrarily given  $\varepsilon$  there is such value  $\delta$  that is a solution of the following equation

$$\frac{1}{\sqrt{D_0 \delta}} \operatorname{ch} \left( \frac{z_0}{\sqrt{D_0 \delta}} \right) \operatorname{cth} \left( \frac{z_0}{\sqrt{D_0 \delta}} \right) = \varepsilon \tag{71}$$

Thus, the function  $g(z, z', t, t')$  is continuous, and hence bounded, in the point  $t = t'$ .

So far as  $G(z, z', t, t') \leq g(z, z', t, t')$  then there is such constant  $K_1$  that as  $|G(z, z', t, t')| \leq K_1$  for  $\forall z, z' \in [0, z_0], \forall t' \in [0, t], \forall t \in [0, \bar{\tau}]$  ( $\bar{\tau} < \infty$ ).

2. First we show boundedness of the function  $|L_s(z)G(z, z', t, t')|$  in all the domain of definition except for the point  $t = t'$ . Taking into account the expression for the Green function (44) and the operator  $L_s(z)$  we obtain

$$L_s(z)G(z, z', t, t') = (D_0 - D_1) \frac{2\theta(t - t')}{z_0} \sum_{i=1}^{n_1} \eta_{i1}(z) \sum_{k=1}^{\infty} y_k^2 e^{-D_0 y_k^2 (t - t')} \sin y_k z \sin y_k z' \tag{72}$$

Consider that  $0 \leq \theta(t - t') \leq 1$  for  $\forall t, t'$  and  $\theta(t - t') = 1$  for  $\forall t, t' \in [0, \bar{\tau}] \cap \{t = t'\}$  as well as

$$\sum_{i=1}^{n_1} \eta_{i1}(x) \leq 1 \quad \text{for } \forall x \tag{73}$$

Then

$$|L_s(z)G(z, z', t, t')| \leq |\bar{g}(z, z', t, t')| = (D_0 - D_1) \frac{2}{z_0} \sum_{k=1}^{\infty} y_k^2 e^{-D_0 y_k^2 (t - t')} \sin y_k z \sin y_k z' \tag{74}$$

As long as the following inequalities are true

$$|D_0 - D_1| \leq d_m, \quad |\sin x| \leq 1 \tag{75}$$

where  $d_m = \max\{D_0; D_1\}$ , then we have

$$|\bar{g}(z, z', t, t')| \leq \frac{2d_m}{z_0} \sum_{k=1}^{\infty} y_k^2 e^{-D_0 y_k^2 (t - t')} \leq \frac{2d_m \pi^2}{z_0^3} \sum_{k=1}^{\infty} k^2 e^{-D_0 \pi^2 |t - t'| k / z_0^2} \tag{76}$$

With provision for  $\sum_{k=1}^{\infty} k^2 x^k = \frac{x(x+1)}{(1-x)^3}$  for  $\forall x, |x| < 1$  [22] and  $e^{-x} < 1$  for  $\forall x > 0$  and also  $D_0 \pi^2 |t - t'| z_0^{-2} > 0$  for  $\forall t, t' \in [0; \bar{\tau}] \cap \{t = t'\}$ , we obtain

$$|\bar{g}(z, z', t, t')| \leq \frac{2d_m \pi^2 e^{-\bar{d}(t-t')} (1 + e^{-\bar{d}|t-t'})}{z_0^3 (1 - e^{-\bar{d}|t-t'})^3} \leq \frac{2d_m \pi^2}{z_0^3} \frac{1 + e^{-\bar{d}|t-t'|}}{(1 - e^{-\bar{d}|t-t'})^3} \tag{77}$$

where  $\bar{d} = D_0 \pi^2 / z_0^2$ .

Since the relation  $|t - t'| = \sigma > 0$  is true then

$$|\bar{g}(z, z', t, t')| \leq \frac{2d_m \pi^2}{z_0^3} \frac{1 + e^{-\bar{d}\sigma}}{(1 - e^{-\bar{d}\sigma})^3} = K_2 \tag{78}$$

To show the boundedness of the function  $\bar{g}(z, z', t, t')$  in the point  $t' = t$ , we use the property of the boundedness of a continuous function on a closed interval. By the definition the function  $\bar{g}(z, z', t, t')$  is continuous with respect to time in the point  $t' = t$ , if for  $\forall \varepsilon > 0 \exists \delta > 0$  such that from the condition  $\forall t, |t - t'| < \delta$  results in the inequality  $|\bar{g}(z, z', t, t') - \bar{g}(z, z', t, t)| < \varepsilon$ .

From the relation (68) we obtain

$$\sum_{k=1}^{\infty} \frac{k^4}{k^2 + a^2} \cos kx = \frac{\pi a^3}{2} \frac{\text{ch}(\pi - x)a}{\text{sh} \pi a} \tag{79}$$

Taking into account (79) and the inequalities (69) and (75) we have

$$\begin{aligned} |\bar{g}(z, z', t, t') - \bar{g}(z, z', t, t)| &< \frac{d_m}{z_0} \left| \sum_{k=1}^{\infty} y_k^2 \cos y_k (z - z') \left[ \frac{1}{1 + D_0 y_k^2} - 1 \right] \right| + \\ + \frac{d_m}{z_0} \left| \sum_{k=1}^{\infty} y_k^2 \cos y_k (z + z') \left[ \frac{1}{1 + D_0 y_k^2} - 1 \right] \right| &\leq \frac{d_m}{2(\sqrt{D_0 \delta})^3} \text{ch} \left( \frac{z_0}{\sqrt{D_0 \delta}} \right) \times \\ &\times \text{cth} \left( \frac{z_0}{\sqrt{D_0 \delta}} \right) = \varepsilon \end{aligned} \tag{80}$$

That is, for the arbitrary number  $\varepsilon$  given beforehand there is such  $\delta$  that is determined from the equation (80). Therefore, by definition, the function  $\bar{g}(z, z', t, t')$  is continuous in the point  $t' = t$  and thus bounded. Then, the function  $L_s(z)G(z, z', t, t')$  is also bounded on any interval from the domain of definition.

3. Show that the action of the operator  $L_s(z)$  on the solution of the homogeneous initial-boundary value problem  $J_0(z, t)$  gives the function bounded in whole domain of definition. With a provision for the expressions  $L_s(z)J_0(z, t) = (D_1 - D_0) \sum_{i=1}^{n_1} \eta_{i1}(z) \frac{\partial^2 J_0(z, t)}{\partial z^2}$  and (39) we have

$$L_s(z)J_0(z, t) = (D_1 - D_0) \frac{2J_*}{z_0} \sum_{i=1}^{n_1} \eta_{i1}(z) \sum_{n=1}^{\infty} e^{-D_0 \xi_n^2 t} \left( \xi_n + \frac{(-1)^n c_* D_0 \xi_n^2}{J_*} \right) \sin \xi_n z \tag{81}$$

Taking into account (73) and (75) we obtain

$$\begin{aligned}
 |L_s(z)J_0(z,t)| &\leq \frac{2J_*d_m}{z_0} \left| \sum_{n=1}^{\infty} e^{-D_0\xi_n^2 t} \left( \xi_n + \frac{(-1)^n c_* D_0 \xi_n^2}{J_*} \right) \sin \xi_n z \right| \leq \\
 &\leq \frac{2J_*d_m}{z_0} \left( \left| \sum_{n=1}^{\infty} e^{-D_0\xi_n^2 t} \xi_n \sin \xi_n z \right| + \left| \sum_{n=1}^{\infty} e^{-D_0\xi_n^2 t} \frac{(-1)^n c_* D_0 \xi_n^2}{J_*} \sin \xi_n z \right| \right) \tag{82}
 \end{aligned}$$

As long as the following equality is true [22]

$$\sum_{k=0}^{\infty} \frac{2k+1}{(2k+1)^2+a^2} \sin(2k+1)x = \frac{\pi}{4} \operatorname{ch}(\pi-2x) \operatorname{asech} \frac{a\pi}{2} \tag{83}$$

as well as the known limit [23]

$$\lim_{|z| \rightarrow \infty} z^\alpha e^{-z} = 0 \quad \text{under } \alpha \equiv \text{const} \tag{84}$$

then, by the Cauchy criterion, the series  $\sum_{n=1}^{\infty} \xi_n^2 e^{-D_0\xi_n^2 t}$  is absolutely convergent. By the Weierstrass criterion of uniform convergence of series, the series  $\sum_{n=1}^{\infty} (-1)^n \xi_n^2 e^{-D_0\xi_n^2 t} \sin \xi_n z$  is absolutely and uniformly convergent, and hence the sequence of its partial sums is also absolutely and uniformly convergent. A convergent sequence is bounded in metric space, then, the series  $\sum_{n=1}^{\infty} (-1)^n \xi_n^2 e^{-D_0\xi_n^2 t} \sin \xi_n z \leq U < \infty$  is bounded. With a provision for (69) and the equality  $\operatorname{ch}^2 t - \operatorname{sh}^2 t = 1$  [23] the following estimation is true

$$\begin{aligned}
 |L_s(z)J_0(z,t)| &\leq \frac{2J_*d_m}{z_0} \left( \frac{z_0}{2D_0t} \operatorname{sech} \frac{z_0}{\sqrt{D_0t}} + \frac{c_* D_0}{J_*} \sum_{n=1}^{\infty} \left| (-1)^n \xi_n^2 e^{-D_0\xi_n^2 t} \right| \right) \leq \\
 &\leq \frac{2J_*d_m}{z_0} \left( \frac{z_0}{2D_0t} \operatorname{sech} \frac{z_0}{\sqrt{D_0t}} + \frac{c_* D_0}{J_*} \sum_{n=1}^{\infty} \xi_n^2 e^{-D_0\xi_n^2 t} \right) \leq \tag{85} \\
 &\leq \frac{2J_*d_m}{z_0} \left( \frac{z_0}{2D_0t} \operatorname{sech} \frac{z_0}{\sqrt{D_0t}} + \frac{c_* D_0}{J_*} U \right) = K_3
 \end{aligned}$$

Thus the Statement is proved.

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