ADJOINT CLASSES OF FUNCTIONS IN THE H_1 SENSE

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(Received January 9, 2004)

Abstract. Using the concept of the H_1-integral, we consider a similarly defined Stieltjes integral. We prove a Riemann-Lebesgue type theorem for this integral and give examples of adjoint classes of functions.

Keywords: Stieltjes integral, Kurzweil integral, Henstock integral, H_1-integral, Riemann-Lebesgue theorem, variational measure, adjoint classes

MSC 2000: 26A39

1. Preliminaries

Symbols $|E|$, χ_E, $\operatorname{int} E$, $\operatorname{cl} E$, $\operatorname{fr} E$ denote the Lebesgue outer measure, the characteristic function, the interior, the closure, and the boundary of a set $E \subset \mathbb{R}$, respectively. If $f\colon E \to \mathbb{R}$ and $A \subset E$ is nonvoid, then $\omega_f(A)$, $f \restriction A$ denote the oscillation of f on A and the restriction of f to A, respectively. We write $\mathscr{D}_f$ for the set of points at which f is discontinuous. We say that f is Baire*1 if for every set $A \subset E$, closed in E, there is a portion $I \cap A \neq \emptyset$ of A such that $f \restriction (I \cap A)$ is continuous. A *figure* means a union of finitely many intervals.

Let $\langle a, b\rangle$ be a nondegenerate compact interval. By a *division* in $\langle a, b\rangle$ we understand any finite collection $\mathscr{P}$ of pairs (I, x) (so-called *tagged intervals*), where I is a compact subinterval of $\langle a, b\rangle$ and its *tag* $x \in I$, such that for all $(I, x), (J, y) \in \mathscr{P}$, if $(I, x) \neq (J, y)$, then the intervals I and J are nonoverlapping. (In papers [8], [12] we used the name *partial tagged partition* instead.) If δ is a *gauge* on $\langle a, b\rangle$, i.e., $\delta\colon \langle a, b\rangle \to (0, \infty)$, then we say that $\mathscr{P}$ is *δ-fine*, if $I \subset (x - \delta(x), x + \delta(x))$ for every $(I, x) \in \mathscr{P}$. We say that $\mathscr{P}$ is *anchored* (contained) in a set E if $x \in E$ ($I \subset E$ respectively) for every $(I, x) \in \mathscr{P}$. If $\bigcup_{(I,x)\in\mathscr{P}} I = \langle a, b\rangle$, then the division $\mathscr{P}$ is called a *partition* of $\langle a,b\rangle$. It will be very useful to write for two divisions: $\mathscr{P} \sqsupseteq \mathscr{R}$, if for each $(I,x) \in \mathscr{P}$ one has $I \subset J$ for some $(J,y) \in \mathscr{R}$.

Let $G\colon \langle a,b\rangle \to \mathbb{R}$. When $I = \langle c,d\rangle \subset \langle a,b\rangle$, by $\Delta G(I)$ we mean the increment $G(d) - G(c)$. Let $\mathscr{P}$ be any collection of pairs (I,x), $I \subset \langle a,b\rangle$, $x \in \langle a,b\rangle$. For $f\colon \langle a,b\rangle \to \mathbb{R}$ we write

$$\sigma_G(\mathscr{P},f) = \sum_{(I,x)\in\mathscr{P}} f(x)\cdot\Delta G(I), \quad |\sigma_G|(\mathscr{P},f) = \sum_{(I,x)\in\mathscr{P}} |f(x)\cdot\Delta G(I)|.$$

Also, $\Delta G(\mathscr{P}) = \sigma_G(\mathscr{P},1)$, $|\Delta|G(\mathscr{P}) = |\sigma_G|(\mathscr{P},1)$.

By $|E|_G$ we mean the *variational measure of* $E \subset \mathbb{R}$ induced by G, see [15]; i.e.,

$$|E|_G = \inf_{\delta} \sup_{\mathscr{P}} |\Delta|G(\mathscr{P}),$$

where sup is taken over all δ-fine divisions $\mathscr{P}$ anchored in E, and inf is taken over all gauges δ. The family $\mathfrak{I}_G$ of subsets of $\langle a,b\rangle$ is defined as follows

$$E \in \mathfrak{I}_G \quad \text{if there exists an } A \in \mathscr{F}_\sigma,\ |A|_G = 0,\ E \subset A;$$

$\mathfrak{I}_{\mathrm{id}} = \mathfrak{I}$, $\mathrm{id}(x) = x$. We will write that a condition holds G-almost everywhere if the exceptional set E has $|E|_G = 0$.

Assume that G is of bounded variation. Then the variational measure $|\cdot|_G$ coincides with the outer measure induced by the ordinary variation of G on open intervals. In this case, a set is called G-measurable if it is measurable with respect to this outer measure.

For notions of AC_*, VB_*, VBG_*, Lusin's $\mathscr{N}$ condition, and their main properties, we refer the reader to [10].

2. The H_1-integral

Notion of the H_1-integral was introduced by Garces, Lee and Zhao in [6]. This concept was based on a modification of the Kurzweil-Henstock integral.

Definition 2.1. We say that f is H_1-integrable to $\mathbf{I}$, if there exists a gauge δ with the following property: for every $\varepsilon > 0$ one can divide $\langle a,b\rangle$ into nonoverlapping intervals $I_1,\dots,I_n$ such that for any δ-fine partitions $\pi_1,\dots,\pi_n$ of $I_1,\dots,I_n$ respectively, we have

$$\left|\sum_{i=1}^{n} \sigma_{\mathrm{id}}(\pi_i,f) - \mathbf{I}\right| < \varepsilon.$$

After the original paper [6], a few further publications appeared [4], [5], [7], [8], [12], [13], and the H_1-integral is already quite thoroughly investigated. Since the H_1-integral is in fact a gauge integral (with the only difference in defining the limit of integral sums in slightly stronger terms), its theory helps to understand better the influence of gauges on Riemann-type integration. Let us sketch the main properties of the H_1-integral. First, every Riemann integrable function is H_1-integrable, and every H_1-integrable function is Kurzweil-Henstock integrable, but the converse statements are not true [6], [12]. There are Lebesgue integrable functions not integrable in the H_1 sense [12], and there are H_1-integrable functions not integrable in the sense of Lebesgue (this is so because the H_1-integral is not absolute) [4], [12]. There is a Kurzweil-Henstock integrable function equal almost everywhere to no H_1-integrable one [8], but every Kurzweil-Henstock integrable function can be written as the sum of a Lebesgue integrable one and an H_1-integrable one [13]. A controlled convergence theorem for the H_1-integral was proved in [5]. However, it is the only convergence theorem which is known for this integral. The Beppo Levi (montone convergence) [12], the Lebesgue (dominated convergence) [12], and even the uniform convergence [7] theorems do not hold. Not every derivative is H_1-integrable, but every derivative is the limit of a uniformly convergent sequence of H_1-integrable functions [8].

A substantial advance was made in the paper [8], where the following Riemann-Lebesgue type theorem for the H_1-integral was obtained, cf. Corollary 3.5 there.

Theorem 2.2. *A function $f\colon \langle a,b\rangle \to \mathbb{R}$ is H_1-integrable if and only if it is Kurzweil-Henstock integrable and*

(1) *there exists an $E \in \mathfrak{I}$ such that $f \restriction (\langle a,b\rangle \setminus E)$ is Baire^{*1} in its domain.*

In the present paper we generalize this result, giving a characterization for Stieltjes H_1-integrable functions. This is a strong generalization, given in terms of Thomson's variational measure. It is not our purpose to present a complete theory of the Stieltjes H_1-integral, which imitates (at least for integrators that are continuous and of bounded variation) the theory already known. The way we lead the reader in the third section, seems to be the shortest way to the R-L type Theorem 3.17. This theorem is not the only goal of our work. We will use it to indicate examples of adjoint classes of functions in the H_1 sense (the fourth section).

3. The main result

Let $f, G\colon \langle a,b\rangle \to \mathbb{R}$.

Definition 3.1. We say that f is Kurzweil-Henstock integrable to $\mathbf{I}$ with respect to G, if for every $\varepsilon > 0$ one can find a gauge δ such that for any δ-fine *partition* π *of* $\langle a,b\rangle$ we have

$$|\sigma_G(\pi, f) - \mathbf{I}| < \varepsilon.$$

Definition 3.2. We say that f is H_1-integrable to $\mathbf{I}$ with respect to G, if there exists a gauge δ with the following property: for every $\varepsilon > 0$ one can divide $\langle a,b\rangle$ into nonoverlapping intervals $I_1, \ldots, I_n$ such that for any δ-fine partitions $\pi_1, \ldots, \pi_n$ of $I_1, \ldots, I_n$ respectively, we have

$$\left|\sum_{i=1}^{n} \sigma_G(\pi_i, f) - \mathbf{I}\right| < \varepsilon.$$

Integrals $\mathbf{I}$ will be denoted by $(\mathrm{H})\int_a^b f\,\mathrm{d}G$ and $(\mathrm{H}_1)\int_a^b f\,\mathrm{d}G$ respectively, or briefly by $\int_a^b f\,\mathrm{d}G$. Infinite ($\pm\infty$) values of $\int_a^b f\,\mathrm{d}G$ are defined in a standard way. The property in Definition 3.2 can be reformulated using the relation $\sqsupseteq$ as follows: for every $\varepsilon > 0$ one can find a partition π_ε of $\langle a,b\rangle$ such that for any δ-fine partition $\pi \sqsupseteq \pi_\varepsilon$, we have $|\sigma_G(\pi, f) - \mathbf{I}| < \varepsilon$.

Lemma 3.3 (Saks-Henstock lemma). *Let f be Kurzweil-Henstock integrable with respect to G, and let F be the indefinite integral of f. Assume that the gauge δ is appropriate for ε in the sense of Definition* 3.1. *Then, for any δ-fine division $\mathscr{P}$ in $\langle a,b\rangle$, we have*

$$|\sigma_G(\mathscr{P}, f) - \Delta F(\mathscr{P})| \leqslant \varepsilon.$$

For the H_1-integral the following version of the Saks-Henstock lemma holds.

Lemma 3.4. *Let f be H_1-integrable with respect to G using gauge δ, and let F be the indefinite integral of f. Assume that intervals $I_1, \ldots, I_n$ are appropriate for ε in the sense of Definition* 3.2. *Then, for any δ-fine divisions $\mathscr{P}_1, \ldots, \mathscr{P}_n$ in $I_1, \ldots, I_n$ respectively, we have*

$$\left|\sum_{i=1}^{n} (\sigma_G(\mathscr{P}_i, f) - \Delta F(\mathscr{P}_i))\right| \leqslant \varepsilon.$$

The next lemma is a corollary of Theorem 43.1 in [15].

Lemma 3.5. *For each $E \subset \langle a, b\rangle$ one has $|G(E)| \leqslant |E|_G$.*

Lemma 3.6. *Let a set D be closed. Suppose that G is AC_* on D and $|D|_G = 0$. Then for each $\varepsilon > 0$ there is a partition π_ε such that for every division $\mathscr{P} \sqsupseteq \pi_\varepsilon$ anchored in D, one has $|\Delta|G(\mathscr{P}) < \varepsilon$.*

Proof. We can assume that D is nowhere dense and that $a, b \in D$. First, let $|D| = 0$. Let (a_i, b_i), $i = 1, 2, 3, \ldots$, be intervals contiguous to D in $\langle a, b\rangle$. Take an $\varepsilon > 0$. Since $D \cap \mathscr{D}_G = \emptyset$, there are intervals $\langle c_i, d_i\rangle \subset (a_i, b_i)$ such that

$$\sum_{i=1}^{\infty} (\omega_G(\langle a_i, c_i\rangle) + \omega_G(\langle d_i, b_i\rangle)) < \frac{\varepsilon}{2}. \tag{2}$$

Also, there is an $\eta > 0$ such that

$$\sum_j |J_j| < \eta \quad \Rightarrow \quad \sum_j \omega_G(J_j) < \frac{\varepsilon}{4} \tag{3}$$

for each family $\{J_j\}_j$ of nonoverlapping intervals with endpoints in D. One can find an N such that

$$\left| \langle a, b\rangle \setminus \bigcup_{i=1}^{N} (a_i, b_i) \right| < \eta. \tag{4}$$

Complete the division $\{(\langle a_i, c_i\rangle, a_i), (\langle d_i, b_i\rangle, b_i)\}_{i=1}^N$ to any partition π_ε of $\langle a, b\rangle$. Consider a division $\mathscr{P} \sqsupseteq \pi_\varepsilon$ anchored in D. Let

$$\mathscr{P}' = \{(I, x) \in \mathscr{P} \colon \ I \subset \langle a_i, c_i\rangle \cup \langle d_i, b_i\rangle, \ i = 1, \ldots, N\}.$$

By (2), $|\Delta|G(\mathscr{P}') < \varepsilon/2$, by (3) and (4), $|\Delta|G(\mathscr{P} \setminus \mathscr{P}') < \varepsilon/2$. Thus, $|\Delta|G(\mathscr{P}) < \varepsilon$.

Now, let $|D|$ be arbitrary. Take a homeomorphism T of $\mathbb{R}$ onto $\mathbb{R}$ such that $D = T(P)$, $|P| = 0$. Since $\Delta G((T(I)) = \Delta(GT)(I)$ for each interval I, we have $|P|_{GT} = |T(P)|_G = 0$. So, by Lemma 3.5, the composition GT satisfies $\mathscr{N}$ on P. Since the set P is closed and GT is VB_* on P, it is AC_* on P. We can apply the first part of the proof to find a partition $\pi_\varepsilon = \{(I_i, x_i)\}_{i=1}^n$ such that $|\Delta|GT(\mathscr{P}) < \varepsilon$ for every division $\mathscr{P} \sqsupseteq \pi_\varepsilon$ anchored in P. One sees that the partition $\{(T(I_i), T(x_i))\}_{i=1}^n$ satisfies the requirement. $\square$

Lemma 3.7. *Let G be* VBG$_*$ *and $E \in \mathfrak{I}_G$. Then every function f is* H$_1$-*integrable with respect to G on E.*

Proof. Let $E = \bigcup_{n=1}^{\infty} E_n$, where G is VB$_*$ on E_n and $|\mathrm{cl}\, E_n|_G = 0$, $n = 1, 2, 3, \ldots$. We can assume that E_n's are pairwise disjoint and that every restriction $f \restriction E_n$ is bounded. G is continuous at each point of $\mathrm{cl}\, E_n$ and satisfies $\mathscr{N}$ on this set (Lemma 3.5). So, G is AC$_*$ on $\mathrm{cl}\, E_n$. Fix an $\varepsilon > 0$. By Lemma 3.6 we can find a partition $\pi_\varepsilon^{(n)}$ of $\langle a, b\rangle$ such that for all divisions $\mathscr{P} \sqsupseteq \pi_\varepsilon^{(n)}$, anchored in $\mathrm{cl}\, E_n$, one has $|\Delta| G(\mathscr{P}) < \varepsilon/M$, where M is an upper bound of $|f| \restriction E_n$. Thus, for such a $\mathscr{P}$,

$$|\sigma_G|(\mathscr{P}, f\chi_{E_n}) \leqslant |\sigma_G|(\mathscr{P}, M) \leqslant M\frac{\varepsilon}{M} = \varepsilon. \tag{5}$$

This means that f is H$_1$-integrable with respect to G on E_n.

For each n there exists a gauge δ_n such that for all δ_n-fine divisions $\mathscr{P}$,

$$|\sigma_G(\mathscr{P}, f\chi_{E_n})| < \frac{1}{2^n}; \tag{6}$$

this comes from the Kurzweil-Henstock integrability of $f\chi_{E_n}$ with respect to G, and from Lemma 3.3. Put $\delta(x) = \delta_n(x)$ for $x \in E_n$, arbitrary outside of E. There is an N such that $1/2^N < \varepsilon/2$. Let a partition π_0 be finer than partitions

$$\pi^{(1)}_{\varepsilon/2N}, \pi^{(2)}_{\varepsilon/2N}, \ldots, \pi^{(N)}_{\varepsilon/2N}.$$

Consider a δ-fine partition $\pi \sqsupseteq \pi_0$ and denote $\mathscr{P}_n = \{(I, x) \in \pi \colon x \in E_n\}$, $n \in \mathbb{N}$. By (5) and (6),

$$\begin{aligned} |\sigma_G(\pi, f\chi_E)| &\leqslant \sum_{n=1}^{N} |\sigma_G|(\mathscr{P}_n, f\chi_{E_n}) + \sum_{n=N+1}^{\infty} |\sigma_G(\mathscr{P}_n, f\chi_{E_n})| \\ &< N\frac{\varepsilon}{2N} + \sum_{n=N+1}^{\infty} \frac{1}{2^n} = \frac{\varepsilon}{2} + \frac{1}{2^N} < \varepsilon. \end{aligned}$$

□

Lemma 3.8 (Cauchy extension). *Suppose that f is Kurzweil-Henstock integrable with respect to G on $\langle a, b\rangle$, and* H$_1$-*integrable with respect to G on every $\langle c, d\rangle \subset (a, b)$. Then it is* H$_1$-*integrable with respect to G on $\langle a, b\rangle$.*

Proof. Similar to that of Lemma 5.3 in [12]. □

Lemma 3.9 (Harnack extension). *Suppose that a set $D \subset \langle a, b\rangle$ is perfect and*

- *f is Kurzweil-Henstock integrable with respect to G on $\langle a, b\rangle$,*
- *f is H_1-integrable with respect to G on every $\langle c, d\rangle \subset \langle a, b\rangle \setminus D$,*
- *G is VB_* on D,*
- *F is VB_* on D, where F is the indefinite integral of f,*
- *$f \restriction D$ is bounded and G-almost everywhere continuous.*

Then f is H_1-integrable with respect to G on $\langle a, b\rangle$.

Proof. Let $I_1, I_2, \ldots$ be closed intervals contiguous to D in $\langle a, b\rangle$. Define a gauge δ on $\langle a, b\rangle$ so that $(x - \delta(x), x + \delta(x)) \subset I_i$ if $x \in \operatorname{int} I_i$, and so that f is H_1-integrable on I_i's using δ (Lemma 3.8). We can assume that for every δ-fine division $\mathscr{P}$ in I_i one has

$$|\sigma_G(\mathscr{P}, f) - \Delta F(\mathscr{P})| < \frac{1}{2^i}. \tag{7}$$

Take arbitrary $\varepsilon > 0$. Consider the set

$$E_\varepsilon = \{x \in D\colon\ \omega(x) \geqslant \varepsilon\},$$

$\omega(x)$ being the oscillation of $f \restriction D$ at x. The set $E_\varepsilon \subset D$ is closed. Since $|E_\varepsilon|_G = 0$, the integrator G is continuous at each point of E_ε and satisfies the condition $\mathscr{N}$ on E_ε. Thus, G is AC_* on E_ε. In virtue of Lemma 3.6 we can find a closed figure $\bigcup\limits_{j=1}^{m} J_j \supset E_\varepsilon$ such that for each division $\mathscr{P}$, anchored in E_ε and contained in $\bigcup\limits_{j=1}^{m} J_j$, we have $|\Delta|G(\mathscr{P}) < \varepsilon$. Of course, we may assume that E_ε is contained in $O = \operatorname{int} \bigcup\limits_{j=1}^{m} J_j$. As G is VB_* on D, if necessary one can shrink J_j's so that $|\Delta|G(\mathscr{P}) < \varepsilon$ will hold for each divison $\mathscr{P}$ conatined in $\bigcup\limits_{j=1}^{m} J_j$ and anchored D. We can split the set $\langle a, b\rangle \setminus O$ into closed intervals $K_1, \ldots, K_p$ such that $\omega_f(K_k \cap D) < \varepsilon$ for each k.

There exists an N such that

$$\sum_{i=N+1}^{\infty} \Big(\omega_G(I_i) + \omega_F(I_i) + \frac{1}{2^i}\Big) < \varepsilon. \tag{8}$$

For $i \leqslant N$ let π^i be a partition of I_i such that for all δ-fine partitions $\pi \sqsupseteq \pi^i$ of I_i one has

$$|\sigma_G(\pi, f) - \Delta F(I_i)| < \frac{\varepsilon}{N}. \tag{9}$$

Let a partition π_0 of $\langle a, b\rangle$ contain some partitions $\pi^{(i)} \sqsupseteq \pi^i$, $i = 1\ldots, N$, and partitions of all intervals J_j and K_k.

Now, consider two arbitrary δ-fine partitions $\pi_1, \pi_2 \sqsupseteq \pi_0$. (We assume that all tags are at endpoints.) Let $\mathscr{P}_s^i = \{(I,x) \in \pi_s \colon x \in I_i, I \subset I_i\}$, $s = 1,2$, $i \leqslant N$. Notice that $\mathscr{P}_s^i$ is a partition of I_i. Now, divisions $\mathscr{Q}_s = \pi_s \setminus \bigcup_{i=1}^{N} \mathscr{P}_s^i$, $s = 1,2$, will be replaced by some families $\mathscr{R}_s$ according to the recipe that follows.

Denote $\tilde{\mathscr{Q}}_s = \{(I,x) \in \mathscr{Q}_s \colon D \cap \operatorname{int} I \neq \emptyset\}$. If $(I,x) \in \mathscr{Q}_s \setminus \tilde{\mathscr{Q}}_s$, the pair (I,x) is included into $\mathscr{R}_s$. Let $(I,x) \in \tilde{\mathscr{Q}}_s$. Notice that $x \in D$. If $\operatorname{fr} I \subset D$, i.e., if both endpoints of I are in D, we include (I,x) into an auxiliary division $\mathscr{O}_s$. In the opposite case, one of the endpoints of $I = \langle c,d\rangle$, say the left one, belongs to some $\operatorname{int} I_i = (a_i, b_i)$; of course $i > N$. Then we include the pair $(\langle c, b_i\rangle, b_i)$ into $\mathscr{R}_s$, and the pair $(\langle b_i, d\rangle, d)$ into $\mathscr{O}_s$. Similarly for the right endpoint situation. Notice that for all $(I,x) \in \mathscr{O}_s$ we have $\operatorname{fr} I \subset D$. Define

$$\mathcal{K} = \{I \cap J \colon (I,x) \in \mathscr{O}_1,\ (J,y) \in \mathscr{O}_2\},$$

where only nondegenerate intervals $I \cap J$ are considered, and include collections $\{(I \cap J, x) \colon I \cap J \in \mathcal{K}\}$ and $\{(I \cap J, y) \colon I \cap J \in \mathcal{K}\}$ into $\mathscr{R}_1$ and $\mathscr{R}_2$ respectively.

Now, let J be the closure of a compound interval of the set

$$\bigcup_{(I,x)\in\mathscr{O}_s} I \setminus \bigcup_{(I,x)\in\mathscr{O}_{3-s}} I.$$

Notice that, since D is perfect, $\operatorname{int} J$ must miss D. Hence $J = I_l = \langle a_l, b_l\rangle$ for some $l > N$. Choose any $c \in (a_l, b_l)$ and include into $\mathscr{R}_s$ the tagged intervals $(\langle a_l, c\rangle, a_l)$ and $(\langle c, b_l\rangle, b_l)$. We have accomplished the construction of $\mathscr{R}_s$. Clearly, $\mathscr{R}_s$ need not be a division. Notice that the intervals from $\mathscr{R}_s$ and $\mathscr{Q}_s$ form partitions of the same figure. Notice also that

$$\{I \colon (I,x) \in \mathscr{S}_1\} = \{I \colon (I,x) \in \mathscr{S}_2\},$$

where $\mathscr{S}_s = \{(I,x) \in \mathscr{R}_s \colon \operatorname{fr} I \subset D\}$, and that if $(I,x) \in \mathscr{R}_s \setminus \mathscr{S}_s$ then $D \cap \operatorname{int} I = \emptyset$.

From the construction of $\mathscr{R}_s$, we obtain by (8) that

$$|\sigma_G(\mathscr{Q}_s, f) - \sigma_G(\mathscr{R}_s, f)| < 4M \sum_{i=N+1}^{\infty} \omega_G(I_i) < 4M\varepsilon,$$

where M is an upper bound of $|f| \restriction D$. Denoting $\mathscr{T}_s = \{(I,x) \in \mathscr{S}_s \colon I \subset O$, we have $|\sigma_G(\mathscr{T}_s, f)| < M\varepsilon$. Also, $|\sigma_G(\mathscr{S}_1 \setminus \mathscr{T}_1, f) - \sigma_G(\mathscr{S}_2 \setminus \mathscr{T}_2, f)| < \varepsilon W$, where W comes from the VB_* property of G on D (notice that the divisions $\mathscr{S}_1 \setminus \mathscr{T}_1$ and $\mathscr{S}_2 \setminus \mathscr{T}_2$ are partitions of the same figure; we use $\omega_f(K_k \cap D) < \varepsilon$ here). Moreover, let

$\mathscr{U}_s^i = \{(I,x) \in \mathscr{R}_s \setminus \mathscr{S}_s \colon\ I \subset I_i,\ x \in D\}$, $\mathscr{V}_s^i = \{(I,x) \in \mathscr{R}_s \setminus \mathscr{S}_s \colon\ I \subset I_i,\ x \notin D\}$. Notice that

$$\mathscr{R}_s \setminus \mathscr{S}_s = \bigcup_{i=N+1}^{\infty} (\mathscr{U}_s^i \cup \mathscr{V}_s^i).$$

Every $\mathscr{V}_s^i$ is a δ-fine partition of a subinterval of I_i. Thus, by (7) and (8),

$$\begin{aligned} |\sigma_G(\mathscr{R}_s \setminus \mathscr{S}_s, f)| &\leqslant \sum_{i=N+1}^{\infty} (|\sigma_G(\mathscr{U}_s^i, f)| + |\sigma_G(\mathscr{V}_s^i, f) - \Delta F(\mathscr{V}_s^i)| + |\Delta F(\mathscr{V}_s^i)|) \\ &\leqslant \sum_{i=N+1}^{\infty} \Bigl(2M\omega_G(I_i) + \frac{1}{2^i} + \omega_F(I_i)\Bigr) < (2M+1)\varepsilon. \end{aligned}$$

By (9),

$$\begin{aligned} &|\sigma_G(\pi_1 \setminus \mathscr{Q}_1, f) - \sigma_G(\pi_2 \setminus \mathscr{Q}_2, f)| \\ &\quad \leqslant \Bigl|\sigma_G(\pi_1 \setminus \mathscr{Q}_1, f) - \sum_{i=1}^{N} \Delta F(I_i)\Bigr| + \Bigl|\sigma_G(\pi_2 \setminus \mathscr{Q}_2, f) - \sum_{i=1}^{N} \Delta F(I_i)\Bigr| \\ &\quad \leqslant \sum_{s=1}^{2} \sum_{i=1}^{N} |\sigma_G(\mathscr{P}_s^i, f) - \Delta F(I_i)| < 2N\frac{\varepsilon}{N} = 2\varepsilon. \end{aligned}$$

Summing these estimates, we obtain

$$\begin{aligned} &|\sigma_G(\pi_1, f) - \sigma_G(\pi_2, f)| \\ &\quad \leqslant \sum_{s=1}^{2} (|\sigma_G(\mathscr{Q}_s, f) - \sigma_G(\mathscr{R}_s, f)| + |\sigma_G(\mathscr{T}_s, f)| + |\sigma_G(\mathscr{R}_s \setminus \mathscr{S}_s, f)|) \\ &\qquad + |\sigma_G(\mathscr{S}_1 \setminus \mathscr{T}_1, f) - \sigma_G(\mathscr{S}_2 \setminus \mathscr{T}_2, f)| + |\sigma_G(\pi_1 \setminus \mathscr{Q}_1, f) - \sigma_G(\pi_2 \setminus \mathscr{Q}_2, f)| \\ &\quad < 8M\varepsilon + 2M\varepsilon + 2(2M+1)\varepsilon + W\varepsilon + 2\varepsilon. \end{aligned}$$

Thus, the Cauchy Criterion for the H_1-integral is fulfilled for f. □

The following lemma was proved in [8], Lemma 3.1.

Lemma 3.10. *Let $E = \bigcup_{n=1}^{\infty} E_n$ be a $\mathscr{G}_\delta$ set and $f\colon\ E \to \mathbb{R}$. If the sequence $(E_n)_n$ is ascending and the restriction $f \restriction E_n$ is continuous for each n, then there exists an open interval J such that $E \cap J \neq \emptyset$ and the restriction $f \restriction (E \cap J)$ is continuous.*

Remark 3.11. Let $E \subset \mathbb{R}$ and assume that $f\colon\ E \to \mathbb{R}$ is bounded and continuous. Define

$$g(x) = \begin{cases} f(x) & \text{if } x \in E, \\ \liminf\limits_{t \to x,\, t \in E} f(t) & \text{if } x \in \operatorname{cl} E \setminus E. \end{cases}$$

Then g is bounded and $\mathscr{D}_g \subset \operatorname{cl} E \setminus E$.

Lemma 3.12. *Let* $G\colon \langle a,b\rangle \to \mathbb{R}$ *be* VBG_*. *Suppose that a function* f *is Kurzweil-Henstock integrable with respect to* G. *Then the indefinite integral* F, *given by*

$$F(x) = (\mathrm{H})\int_a^x f\,\mathrm{d}G,$$

has the VBG_* *property as well.*

P r o o f. We will use a condition which is equivalent to the VBG_* property: "the variational measure is σ-finite on a co-countable subset of $\langle a,b\rangle$". Suppose that $|D|_G < \infty$. Denote $E_n = \{x \in \langle a,b\rangle\colon |f(x)| \leqslant n\}$, $n = 1,2,\ldots$. We will show that $|D \cap E_n|_F < \infty$ for each n and this will complete the proof.

Let a gauge δ be suitable for $\varepsilon = 1$ in the sense of Definition 3.1. We may suppose that

$$|\Delta| G(\mathscr{P}) < |D \cap E_n|_G + 1$$

for each δ-fine division $\mathscr{P}$ anchored in $D \cap E_n$. Then, by Lemma 3.3, for such a $\mathscr{P}$ one has

$$\begin{aligned}|\Delta| F(\mathscr{P}) &\leqslant \sum_{(I,x)\in\mathscr{P}} |f(x)\cdot \Delta G(I) - \Delta F(I)| + \sum_{(I,x)\in\mathscr{P}} |f(x)|\cdot|\Delta G(I)|\\ &\leqslant 2 + n|D\cap E_n|_G + n.\end{aligned}$$

It means that $|D \cap E_n|_F < \infty$. $\square$

Theorem 3.13. *Let* $f, G\colon \langle a,b\rangle \to \mathbb{R}$ *and let* $G \in \mathrm{VBG}_*$. *Consider the following two assertions:*

(i) f *is Kurzweil-Henstock integrable with respect to* G, *and*

$$\begin{gathered}\text{for each nonempty closed set } D \subset \langle a,b\rangle \text{ one can find an } A \in \mathfrak{I}_G \text{ and}\\ \text{an interval } I \text{ with } I \cap D \setminus A \neq \emptyset \text{ such that } f \restriction (I \cap D \setminus A) \text{ is continuous;}\end{gathered} \tag{10}$$

(ii) f *is* H_1*-integrable with respect to* G.

One has (i) $\Rightarrow$ (ii). The converse holds if G is continuous.

P r o o f. (i) $\Leftarrow$ (ii) Suppose that f does not satisfy the condition (10). We will show that f is not H_1-integrable with respect to G. Consider an arbitrary gauge δ on $\langle a,b\rangle$. Let D be a closed subset of $\langle a,b\rangle$ such that for each $A \in \mathfrak{I}_G$, the set of discontinuity points of $f \restriction (D \setminus A)$ is dense in $D \setminus A \neq \emptyset$. Of course $D \notin \mathfrak{I}_G$. Put $D_n = \{x \in D\colon \delta(x) > 1/n\}$, $n \in \mathbb{N}$. In virtue of Lemma 3.10, there exists an n such that

$$C = \{x \in D_n\colon f \restriction D_n \text{ is discontinuous at } x\} \notin \mathfrak{I}_G.$$

For $x \in C$, denote by $\omega(x)$ the oscillation of $f \restriction D_n$ at x; one has $\omega(x) > 0$. Since $C \notin \mathfrak{I}_G$, for some m the set $C_m = \{x \in C\colon\ \omega(x) > 1/m\}$ satisfies $|\operatorname{cl} C_m|_G > M > 0$. Take any π. One can cover $\operatorname{cl} C_m$ by a family $\mathscr{A}$ of nonoverlapping intervals, all of length less than $1/n$, to satisfy

$$\sum_{I\in\mathscr{A}} |\Delta G(I)| > M.$$

If G is continuous we can assume that $C_m \cap \operatorname{int} I \neq \emptyset$ for each $I \in \mathscr{A}$. We can also assume that $I \subset J$ for some $(J, y) \in \pi$. For each $I \in \mathscr{A}$ one can pick an $x_I \in C_m \cap \operatorname{int} I$ and a $y_I \in D_n \cap I$ such that $|f(x_I) - f(y_I)| > 1/m$. Both divisions

$$\mathscr{P}_1 = \{(I, x_I)\}_{I\in\mathscr{A}}, \quad \mathscr{P}_2 = \{(I, y_I)\}_{I\in\mathscr{A}}$$

are δ-fine, moreover $\mathscr{P}_1, \mathscr{P}_2 \sqsupseteq \pi$. We get

$$\sum_{I\in\mathscr{A}} |f(x_I) - f(y_I)||\Delta G(I)| > \frac{M}{m}.$$

M and m were found independently of π, whence Lemma 3.4 is not valid for f using δ. Thus, f is not H_1-integrable with respect to G. (We notice that the (i) $\Leftarrow$ (ii) part of this reasoning follows for any continuous G.)

(i)$\Rightarrow$(ii) Suppose that f is Kurzweil-Henstock but not H_1-integrable with respect to G. Let $P \neq \emptyset$ be the set of all points $x \in \langle a, b\rangle$ such that f is integrable on no neighbourhood of x. Lemma 3.8 implies that P is perfect and that f is integrable on the closure of every interval contiguous to P. Assume that f satisfies the condition (10). There exists a portion $I \cap P$ of P such that for some $A \in \mathfrak{I}_G$ the restriction $f \restriction (I \cap P \setminus A)$ is continuous and bounded, and both the integrator G and the Kurzweil-Henstock Stieltjes indefined integral of f are VB_* on $I \cap P$ (Lemma 3.12). Extend the restriction $f \restriction (I \cap P \setminus A)$ to a g on $I \cap P$, as is described in Remark 3.11. Put $\tilde{f} = g$ on $I \cap P$, $\tilde{f} = f$ otherwise. Since $\mathscr{D}_g \subset A$, by Lemma 3.9, $\tilde{f}$ is H_1-integrable on I with respect to G. Hence by Lemma 3.7, f is H_1-integrable with respect to G on I, a contradiction. $\square$

Remark 3.14. The (i) $\Leftarrow$ (ii) part of Theorem 3.13 holds also if G is *normalized*, i.e., if at each $x \in (a, b)$ there are finite $G(x+), G(x-)$ with $2G(x) = G(x+) + G(x-)$. In this case, one can get $\sum\limits_{I\in\mathscr{A}'} |\Delta G(I)| > M/2$, where $\mathscr{A}' \subset \mathscr{A}$ contains these I which do not miss C_m. The rest of the proof follows with M replaced by $M/2$.

Example 3.15. Let $\mathbb{C}$ be the Cantor ternary set. Define $f, G\colon \langle 0, 1\rangle \to \mathbb{R}$ as follows. Put $f = 0$ on closure of each interval contiguous to $\mathbb{C}$ in $\langle 0, 1\rangle$, 1 otherwise.

Enumerate endpoints of these contiguous intervals as $\{x_n\}_{n=1}^{\infty}$ and define $G(x) = \sum\limits_{n\colon x_n<x} 2^{-n}$ for each $x \notin \{x_n\}_{n=1}^{\infty}$, $G(x_n) = G(x_n-)$ if x_n is right-isolated in $\mathbb{C}$, $G(x_n) = G(x_n+)$ if x_n is left-isolated in $\mathbb{C}$. It is a matter of routine to show that f is Darboux integrable (so H_1-integrable) to 0 with respect to G. Each closed subset $E \subset \mathbb{C}$ with $|E|_G = 0$ is nowhere dense in $\mathbb{C}$ (since $E \cap \{x_n\}_{n=1}^{\infty} = \emptyset$). Hence each $A \in \mathfrak{I}_G$, $A \subset \mathbb{C}$, is of 1st category in $\mathbb{C}$. From Baire Category Theorem $\mathbb{C} \setminus (\{x_n\}_{n=1}^{\infty} \cup A)$ is dense in $\mathbb{C}$ and so $f \restriction (\mathbb{C} \setminus A)$ is everywhere discontinuous. The condition (10) is not fulfilled. (i) $\Leftarrow$ (ii) of Theorem 3.13 fails to hold for all $G \in \mathrm{VBG}_*$, in contrary to what we have announced in [14].

The set $A \in \mathfrak{I}_G$ from Theorem 3.13 can be chosen independently of D.

Lemma 3.16. *The condition* (10) *is equivalent to the following one:*

$$\text{(11)} \qquad \begin{gathered} \textit{there exists a } B \in \mathfrak{I}_G \textit{ such that} \\ \textit{the restriction } f \restriction (\langle a,b\rangle \setminus B) \textit{ is Baire}^*1 \textit{ in its domain;} \end{gathered}$$

i.e., there exists a $B \in \mathfrak{I}_G$ with the property that for every closed set $P \subset \langle a,b\rangle$ with $P \setminus B \neq \emptyset$ we can find an open interval J with $P \cap J \setminus B \neq \emptyset$ such that the restriction $f \restriction (P \cap J \setminus B)$ is continuous.

P r o o f. Repetition of the proof of Lemma 3.4 in [8]. □

By Theorem 3.13 and Lemma 3.16 we obtain the following Riemann-Lebesgue type theorem for the Stieltjes H_1-integral.

Theorem 3.17. *Let $f, G\colon \langle a,b\rangle \to \mathbb{R}$, and let G be continuous and* VBG_*. *The following assertions are equivalent:*

- *the function f is H_1-integrable with respect to G;*
- *the function f is Kurzweil-Henstock integrable with respect to G and there is a $B \in \mathfrak{I}_G$ such that $f \restriction (\langle a,b\rangle \setminus B)$ is Baire*1 in its domain.*

We notice that the above theorem reveals some advantage of the H_1-integral over the Riemann integral. If the integrator G is of bounded variation, then one may give the following Riemann-Lebesgue type theorem (roughly formulated).

Observation 3.18. An f is Riemann integrable with respect to G iff it is G-almost everywhere continuous and bounded outside a closed subfigure of the open figure on which G is interval-wise constant.

Nevertheless, if G is taken from the class VBG_*, a simple R-L type theorem for the Riemann integral seems to be unavailable. One may easily find two continuous VBG_*-functions f and G such that f is Kurzweil-Henstock integrable with respect to G, but not Riemann integrable with respect to G. The H_1-integral allows a quite simple R-L theorem, even for VBG_* integrators.

Problem 3.19. Is Theorem 3.17 true for every continuous integrator G?

4. Adjoint classes

We use the following symbols to denote classes of functions on $\langle a,b\rangle$: C: continuous functions; AC^q: absolutely continuous functions G with $|G'|^q$ summable, $\mathrm{AC}^1=\mathrm{AC}$, AC^∞: Lipschitz functions; VB: functions with bounded variation; $\mathscr{B}_1^*$: Baire*1 functions; R: Riemann integrable functions; H_1: H_1-integrable functions, $|\mathrm{H}_1|^q$: H_1-integrable functions f with $|f|^q$ also H_1-integrable, $|\mathrm{H}_1|^1=|\mathrm{H}_1|$, $|\mathrm{H}_1|^\infty$: essentially bounded H_1-integrable functions; B: bounded functions; $\tilde{\mathrm{C}}$: bounded functions f with countable set $\mathscr{D}_f$; $\bar{\mathrm{C}}$: interval-wise constant functions ($f\in\bar{\mathrm{C}}$ if $\langle a,b\rangle$ can be divided into finite number of intervals $I_1,\dots,I_n$, such that the restriction $f\restriction(\operatorname{int} I_i)$ is constant for each i).

Let T be a Stieltjes integration process on an interval. We say that classes $\mathscr{A}$ (of integrands) and $\mathscr{B}$ (of integrators) are *adjoint* [1] in the T sense (abbr. $\mathscr{A}\overset{\mathrm{T}}{*}\mathscr{B}$), if

- for each $f\in\mathscr{A}$ and $G\in\mathscr{B}$, the function f is T-integrable with respect to G;
- for each $f\notin\mathscr{A}$ (each $G\notin\mathscr{B}$), there is a $G\in\mathscr{B}$ (an $f\in\mathscr{A}$ respectively) such that f is T-nonintegrable with respect to G.

Hanxiang Chen in his papers [1], [2], [3] gave several pairs of adjoint classes. He considered the Riemann integral [1], [2], the Young (Ross-Riemann) integral [2], see also [9], [11], and the Lebesgue integral [3]. We mention below only Riemann pairs:

$$\text{(i)}\ \ \mathrm{C}\overset{\mathrm{R}}{*}\mathrm{VB}\ \ \text{(a well known one)},\qquad \text{(ii)}\ \ \mathrm{R}\overset{\mathrm{R}}{*}\mathrm{AC},\qquad \text{(iii)}\ \ \tilde{\mathrm{C}}\overset{\mathrm{R}}{*}\mathrm{VB}\cap\mathrm{C}.$$

Using the above proved Riemann-Lebesgue type theorem, we point two pairs of adjoint classes in the H_1 sense (Theorems 4.3 and 4.5). We start with simple lemmas.

Lemma 4.1. *Suppose that $G\in\mathrm{AC}$. Then the Kurzweil-Henstock Stieltjes integral $(\mathrm{H})\int_a^b f\,\mathrm{d}G$ exists if and only if the Kurzweil-Henstock integral $(\mathrm{H})\int_a^b fG'$ exists. Moreover, these two integrals are equal.*

P r o o f. Let E be the set of points of $\langle a,b\rangle$ at which G is not differentiable, $|E|=0$; we may assume that $G'=0$ on E. Denote $D_n=\{x\in\langle a,b\rangle\colon n-1\leqslant$

$|f(x)| < n\}$, $\bigcup\limits_{n=1}^{\infty} D_n = \langle a,b\rangle$. Fix an $\varepsilon > 0$. Since G is absolutely continuous, there is a gauge δ on E such that for each δ-fine division $\mathscr{P}$ anchored in $E \cap D_n$ the inequality $|\Delta| G(\mathscr{P}) < \varepsilon/n2^n$ holds. For an $x \in D_n \setminus E$ there is a number $\delta(x) > 0$ such that for every interval $I \ni x$, $|I| < \delta(x)$, one has $|\Delta G(I) - G'(x)|I|| \leqslant \varepsilon |I|/n$. Consider a δ-fine partition π of $\langle a,b\rangle$. Denote $\mathscr{P}_E = \{(I,x) \in \pi \colon x \in E\}$, $\mathscr{P}_n = \{(I,x) \in \pi \colon x \in D_n\}$, $n = 1,2,\ldots$. The following estimate completes the proof.

$$\begin{aligned}
|\sigma_G(\pi,f) - \sigma_{\mathrm{id}}(\pi, fG')| &\leqslant \sum_{n=1}^{\infty} \sum_{(I,x)\in \mathscr{P}_n \setminus \mathscr{P}_E} |f(x)| \cdot |\Delta G(I) - G'(x)|I|| \\
&\quad + \sum_{n=1}^{\infty} |\sigma_G(\mathscr{P}_E \cap \mathscr{P}_n, f)| \\
&\leqslant \sum_{n=1}^{\infty} \sum_{(I,x)\in \mathscr{P}_n \setminus \mathscr{P}_E} n \cdot \frac{\varepsilon}{n}|I| + \sum_{n=1}^{\infty} n \cdot |\Delta| G(\mathscr{P}_E \cap \mathscr{P}_n) \\
&< \varepsilon(b-a) + \sum_{n=1}^{\infty} n \frac{\varepsilon}{n2^n} = \varepsilon(b-a+1).
\end{aligned}$$

□

Lemma 4.2. *Suppose that a function $G \colon \langle a,b\rangle \to \mathbb{R}$ is continuous and of bounded variation, a set $E \subset \langle a,b\rangle$ is closed. There exists a set E_1 such that both E_1 and $E \setminus E_1$ have positive measure $|\cdot|_G$ in every portion of E, which has positive measure $|\cdot|_G$.*

P r o o f. We may assume that $c = |E|_G > 0$. Define

$$F(x) = |\langle a,x\rangle \cap E|_G, \quad x \in \langle a,b\rangle.$$

F is continuous, because G is so. Let C_1 be a perfect nowhere dense subset of the interval $\langle 0,c\rangle$ with measure $c/2$. We proceed by induction. Having defined a perfect nowhere dense set C_n, let $\{I_i^{(n)}\}_{i=1}^{\infty}$ be intervals contiguous to C_n in $\langle 0,c\rangle$. In every $I_i^{(n)}$ choose a closed nowhere dense subset $C_i^{(n)}$ with $|C_i^{(n)}| = |I_i^{(n)}|/2^n$. Define $C_{n+1} = C_n \cup \bigcup\limits_{i=1}^{\infty} C_i^{(n)}$ and $C = \bigcup\limits_{n=1}^{\infty} C_n$. The sets C and $\langle 0,c\rangle \setminus C$ have positive measure in every subinterval of $\langle 0,c\rangle$. Put $E_1 = F^{-1}(C) \cap E$. It is seen that E_1 fulfils the requirement. □

Theorem 4.3. *Let $1/p + 1/q = 1$, $\infty \geqslant p, q \geqslant 1$. Then*

$$|\mathrm{H}_1|^p \overset{\mathrm{H}_1}{*} (\mathrm{AC}^q + \overline{\mathrm{C}}).$$

P r o o f. (a) Consider any $f \in |\mathrm{H}_1|^p$. Let $G \in \mathrm{AC}^q$. Since fG' is Kurzweil-Henstock integrable, from Lemma 4.1 we get

$$(\mathrm{H})\int_a^b fG' = (\mathrm{H})\int_a^b f\,\mathrm{d}G,$$

so f is Kurzweil-Henstock integrable with respect to G. Since f fulfils the condition (1), from Theorem 3.17 we see that f is H_1-integrable with respect to G.

(b) Assume first that $G \in \mathrm{VB} \cap \mathrm{C} \setminus \mathrm{AC}$. Since G does not satisfy condition $\mathscr{N}$, there exists a closed set $E \subset \langle a, b\rangle$ with $|E| = 0$ and $|G(E)| > 0$. From Lemma 3.5 we have $|G(E)| \leqslant |E|_G$, so $|E|_G > 0$. Using Lemma 4.2, divide E into two sets having positive variational measure $|\cdot|_G$ in each portion of E, $E = E_1 \cup E_2$. Define $f = \chi_{E_1}$. The function f belongs to $|\mathrm{H}_1|^p$, but for any $A \in \mathfrak{I}_G$, $f \restriction (E \setminus A)$ is discontinuous everywhere. By Theorem 3.17, f is not H_1-integrable with respect to G. So, assume that $G \in \mathrm{AC} \setminus \mathrm{AC}^q$. There are two cases to consider.

($1 < q < \infty$) We have $(\mathrm{H})\int_a^b |G'|^q = \infty$. There exists a point $\xi \in \langle a, b\rangle$ in whose every neighborhood $|G'|^q$ is nonintegrable. One can find disjoint tagged intervals $\{(I_n, x_n)\}_{n=1}^{\infty}$, with x_n's converging to ξ, such that

$$\sum_{n=1}^{\infty} |G'(x_n)|^q |I_n| = \infty.$$

(We can assume that $|\Delta G(I_n) - G'(x_n)|I_n|| < |I_n|/2^n$.) There exists a sequence $(a_n)_{n=1}^{\infty}$ of positive numbers such that

$$\sum_{n=1}^{\infty} a_n^p < \infty \tag{12}$$

and

$$\sum_{n=1}^{\infty} |G'(x_n)| a_n \sqrt[q]{|I_n|} = \infty.$$

Put $f(x) = a_n |I_n|^{-1/p}$ for $x \in I_n$, 0 otherwise. Then

$$\begin{aligned}\left|(\mathrm{H})\int_a^b f\,\mathrm{d}G\right| &= \left|\sum_{n=1}^{\infty} \frac{a_n}{\sqrt[p]{|I_n|}} \Delta G(I_n)\right| = \left|\sum_{n=1}^{\infty} a_n \sqrt[q]{|I_n|} \frac{\Delta G(I_n)}{|I_n|}\right| \\ &\geqslant \sum_{n=1}^{\infty} |G'(x_n)| a_n \sqrt[q]{|I_n|} - \sum_{n=1}^{\infty} \sqrt[q]{|I_n|} \frac{a_n}{2^n} = \infty.\end{aligned}$$

Thus f is Kurzweil-Henstock- and, in consequence, H_1-nonintegrable with respect to G. One sees this f is Baire*1, hence by (12) it belongs to the class $|\mathrm{H}_1|^p$.

($q = \infty$) There is a point $\xi \in \langle a, b\rangle$ in whose every neighborhood G' is unbounded. One can find disjoint tagged intervals $\{(I_n, x_n)\}_{n=1}^{\infty}$, with $x_n \to \xi$, such that $|\Delta G(I_n) - G'(x_n)|I_n|| \leqslant |I_n|$. Put $f(x) = 1/2^n|I_n|$ for $x \in I_n$, 0 otherwise. Then

$$\left|(\mathrm{H})\int_a^b f\,\mathrm{d}G\right| = \left|\sum_{n=1}^{\infty} \frac{\Delta G(I_n)}{2^n|I_n|}\right| \geqslant \sum_{n=1}^{\infty} \frac{|G'(x_n)|}{2^n} - \sum_{n=1}^{\infty} \frac{1}{2^n} = \infty.$$

On the other hand, f is Baire*1 and

$$(\mathrm{H})\int_a^b |f| = \sum_{n=1}^{\infty} \frac{1}{2^n|I_n|}|I_n| < \infty,$$

whence by Theorem 3.17, $f \in |\mathrm{H}_1|$.

Suppose now that the jump part of $G \in \mathrm{VB}$ does not belong to the class $\overline{\mathrm{C}}$. Then, there exists a countable set $S \subset \langle a, b\rangle$ such that $|\{x\}|_G > 0$ for each $x \in S$. We may assume that $G(x+) \neq G(x-)$ for all $x \in S$. Put

$$f(x) = \begin{cases} 1/(G(x+) - G(x-)) & \text{if } x \in S,\\ 0 & \text{otherwise.}\end{cases}$$

$f \in |\mathrm{H}_1|^p$ and it is an easy exercise to show that f is not Kurzweil-Henstock integrable with respect to G; see Example 2.1 in [11].

(c) Let $f \notin |\mathrm{H}_1|^p$. If $f \notin \mathrm{H}_1$, then take $G = \mathrm{id}$. If $f \in \mathrm{H}_1 \setminus |\mathrm{H}_1|^p$, then $|f|$ is Kurzweil-Henstock nonintegrable with the pth power. From the Riesz theorem we find a function g, absolutely integrable with the qth power, such that

$$(\mathrm{H})\int_a^b fg = (\mathrm{H})\int_a^b fG' = \infty,$$

where $G(x) = (\mathrm{H})\int_a^x g$. Since G is AC^q, by Lemma 4.1 we conclude $\infty = (\mathrm{H})\int_a^b fG' = (\mathrm{H})\int_a^b f\,\mathrm{d}G$. Let us remark that Theorem 4.3 is the correction to the pair (8) from [14]. □

Remark 4.4. Let $f\colon E \to \mathbb{R}$ be discontinuous at every point of E. Then there exists a countable set $C \subset E$, dense in E, such that $f \restriction C$ is discontinuous at every point of C.

Proof. For each interval (u, v) with rational endpoints with $(u, v) \cap E \neq \emptyset$, choose $\xi, \psi \in (u, v) \cap E$ such that $|f(\xi) - f(\psi)| > \frac{1}{2}\omega_f((u, v) \cap E)$. Collecting such ξ and ψ for all (u, v), we obtain the desired set C. □

Theorem 4.5. $\mathscr{B}_1^* \cap \mathrm{B} \overset{\mathrm{H}_1}{*} \mathrm{VB}$.

Proof. (a) First, we need to check that for a given bounded Baire*1 function f and a VB-function G, the integral $(\mathrm{H}_1)\int_a^b f\,\mathrm{d}G$ exists. This is immediate, since $(\mathrm{H})\int_a^b f\,\mathrm{d}G$ exists (f is bounded and G-measurable since Borel) and f is Baire*1 (Theorem 3.13).

(b) Let G be of unbounded variation. There are nonoverlapping intervals $I_1, \ldots$ such that $\sum\limits_{n=1}^{\infty} \Delta G(I_n) = \pm\infty$. Put $f(x) = 1$ on $\bigcup\limits_{n=1}^{\infty} I_n$, 0 otherwise. Of course, $f \in \mathscr{B}_1^* \cap \mathrm{B}$ and $(\mathrm{H})\int_a^b f\,\mathrm{d}G = \pm\infty$.

(c) Suppose f is unbounded. Then there are points $x_n \in \langle a, b\rangle$, $n \in \mathbb{N}$, such that $|f(x_n)| > 2^n$. Put

$$G(x) = \sum_{n\colon x_n < x} \frac{\operatorname{sgn} f(x_n)}{2^n}. \tag{13}$$

G is of bounded variation and f is not H_1-integrable with respect to G. Suppose f is not Baire*1. Then there is a closed set $E \subset \langle a, b\rangle$ such that $\mathscr{D}_{f\restriction E}$ is dense in E. Applying Remark 4.4 to $\mathscr{D}_{f\restriction E}$ (as one can check, $f \restriction (\mathscr{D}_{f\restriction E})$ is discontinuous at each point of $\mathscr{D}_{f\restriction E}$), we get points $x_1, x_2, x_3, \ldots \in E$ such that $S = \{x_n\}_{n=1}^{\infty}$ is dense in E and $f \restriction S$ is discontinuous everywhere. Define an integrator G by the formula (13) for $x \neq x_n$, $n \in \mathbb{N}$, and assume G is normalized; i.e., $2G(x_n) = G(x_n+) + G(x_n-)$. Since for all $X \in \mathfrak{I}_G$ we have $X \cap S = \emptyset$, the restriction $f \restriction (E \setminus X)$ has a dense set of discontinuity points, and thus $f \restriction (E \setminus X)$ is not Baire*1. By Remark 3.14, f is not H_1-integrable with respect to the $G \in \mathrm{VB}$. $\square$

The adjoint pairs from Theorems 4.3 and 4.5 resemble the pairs (ii) and (i), page 13, respectively. We have not been able to find an analogue of the pair (iii).

Problem 4.6. Find a pair of adjoint classes for the H_1-integral, similar to the pair (iii).

References

[1] *H. Chen*: A pair of adjoint classes of Riemann-Stieltjes integrable functions. Real Anal. Exch. *23* (1998), 235–240. zbl
[2] *H. Chen*: Adjoint classes of generalized Stieltjes integrable functions. Real Anal. Exch. *24* (1999), 139–148. zbl
[3] *H. Chen*: Adjoint classes of Lebesgue-Stieltjes integrable functions. Real Anal. Exch. *26* (2001), 421–427. zbl
[4] *I. J. L. Garces, P. Y. Lee*: Cauchy and Harnack extensions for the H_1-integral. Matimyás Mat. *21* (1998), 28–34. zbl
[5] *I. J. L. Garces, P. Y. Lee*: Convergence theorems for the H_1-integral. Taiwanese J. Math. *4* (2000), 439–445. zbl

[6] *I. J. L. Garces, P. Y. Lee, and D. Zhao*: Moore-Smith limits and the Henstock integral. Real Anal. Exch. *24* (1999), 447–455. zbl

[7] *A. Maliszewski, P. Sworowski*: Uniform convergence theorem for the H_1-integral revisited. Taiwanese J. Math. *7* (2003), 503–505. zbl

[8] *A. Maliszewski, P. Sworowski*: A characterization of H_1-integrable functions. Real Anal. Exch. *28* (2003), 93–104. zbl

[9] *K. A. Ross*: Another approach to Riemann-Stieltjes integrals. Am. Math. Mon. *87* (1980), 660–662. zbl

[10] *S. Saks*: Theory of the Integral. G. E. Stechert, New York, 1937. zbl

[11] *Š. Schwabik*: On the relation between Young's and Kurzweil's concept of Stieltjes integral. Cas. Pest. Mat. *98* (1973), 237–251. zbl

[12] *P. Sworowski*: On H_1-integrable functions. Real Anal. Exch. *27* (2002), 275–286. zbl

[13] *P. Sworowski*: Some comments on the H_1-integral. Real Anal. Exch. *29* (2004), 789–797. zbl

[14] *P. Sworowski*: Adjoint classes for generalized Riemann-Stieltjes integrals. 27th Summer Symposium Conference Reports, Opava 2003. Real Anal. Exch. (2003), 41–45.

[15] *B. S. Thomson*: Real Functions. Lecture Notes in Mathematics, Vol. 1170. Springer-Verlag, Berlin, 1985.

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