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THE MAXIMAL CLASS WITH RESPECT TO MAXIMUMS FOR THE FAMILY OF UPPER SEMICONTINUOUS STRONG ŚWIĄTKOWSKI FUNCTIONS

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ABSTRACT. The main goal of this paper is to characterize the maximal class with respect to maximums for the family of upper semicontinuous strong Świątkowski functions.

1. Introduction

We use mostly standard terminology and notation. The letters $\mathbb{R}$ and $\mathbb{N}$ denote the real line and the set of positive integers, respectively. The symbol $\mathrm{I}(a,b)$ denotes the open interval with endpoints a and b. For each $A \subset \mathbb{R}$ we use the symbols $\operatorname{int} A$, $\operatorname{cl} A$, $\operatorname{bd} A$, and $|A|$ to denote the interior, the closure, the boundary, and the outer Lebesgue measure of A, respectively. The Euclidean metric in $\mathbb{R}$ will be denoted by dist.

Let I be a nondegenerate interval and $f\colon I \to \mathbb{R}$. We say that f is a *Darboux function* ($f \in \mathscr{D}$), if it maps connected sets onto connected sets. The symbols $\mathscr{C}(f)$ and $\mathscr{C}^-(f)$ will stand for the set of points of continuity and left-hand continuity of f, respectively. We say that f is a *strong Świątkowski function* [4] ($f \in \acute{\mathcal{S}}_s$), if whenever $\alpha, \beta \in I$, $\alpha < \beta$, and $y \in \mathrm{I}(f(\alpha), f(\beta))$, there is an $x_0 \in (\alpha,\beta) \cap \mathscr{C}(f)$ such that $f(x_0) = y$. The symbols $\mathscr{C}$ and usc denote families

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of all continuous and upper semicontinuous functions, respectively. The function f is *upper semicontinuous strong Świątkowski* ($f \in \acute{\mathcal{S}}_{\mathrm{susc}}$), if it is both upper semicontinuous and strong Świątkowski. (Clearly $\acute{\mathcal{S}}_{\mathrm{susc}} \subset \acute{\mathcal{S}}_s \subset \mathscr{D}$ and both inclusions are proper.) We say that $f \in \mathscr{C}$ if and only if $f[I]$ is a singleton. The symbol $[f = a]$ stands for the set $\{x \in I : \ f(x) = a\}$. Similarly we define the symbols $[f < a]$, $[f \geq a]$, etc.

Let $f \colon \mathbb{R} \to \mathbb{R}$. If $A \subset \mathbb{R}$ and x is a limit point of A, then let

$$\overline{\lim}(f, A, x) = \overline{\lim_{t \to x,\, t \in A}} f(t).$$

Similarly we define $\overline{\lim}(f, A, x^-)$, $\underline{\lim}(f, A, x^+)$, etc. Moreover we write $\overline{\lim}(f, x^-)$ instead of $\overline{\lim}(f, \mathbb{R}, x^-)$, etc. If $\mathscr{L}$ is a family of real functions, then we define the maximal class with respect to maximums for $\mathscr{L}$ as follows:

$$\mathcal{M}_{\max}(\mathscr{L}) = \Big\{ f : \ \underset{g \in \mathscr{L}}{\forall} \max\{f, g\} \in \mathscr{L} \Big\}.$$

It is known that $\mathcal{M}_{\max}(\mathrm{usc}) = \mathrm{usc}$ (see e.g. [2]). In 1971 Farková characterized the maximal class with respect to maximums for the family of Darboux functions, which is equal to the family of Darboux upper semicontinuous functions [1]. In 2003 I proved that $\mathcal{M}_{\max}(\acute{\mathcal{S}}_s) = \mathscr{C}$ ([5]). In this paper we characterize the maximal class with respect to maximums for the family of upper semicontinuous strong Świątkowski functions. It turns out that $\mathcal{M}_{\max}(\acute{\mathcal{S}}_{\mathrm{susc}})$ consist of upper semicontinuous strong Świątkowski functions which satisfied some special conditions. (Theorems 2.4 and 2.5). In particular the maximum of a continuous function and an upper semicontinuous strong Świątkowski function is upper semicontinuous strong Świątkowski (Corollary 2.6).

2. Main results

Lemma 2.1 can be easily proved using [3: Theorem 12].

LEMMA 2.1. *Let $I \subset \mathbb{R}$ be an interval, the function $g \colon I \to \mathbb{R}$, and $x \in I$. If $g \restriction I \cap (-\infty, x] \in \acute{\mathcal{S}}_s$, $g \restriction I \cap (x, \infty) \in \acute{\mathcal{S}}_s$, and $g(x) \in g\big[[x, t] \cap \mathscr{C}(g)\big]$ for each $t \in (x, \sup I)$, then $g \in \acute{\mathcal{S}}_s$.*

The proof of Lemma 2.2 we can find in [6: Lemma 3.4].

LEMMA 2.2. *Assume that $F \subset C$ are closed and $\mathcal{J}$ is a family of components of $\mathbb{R} \setminus C$ such that $C \subset \operatorname{cl} \bigcup \mathcal{J}$. There is a family $\mathcal{J}' \subset \mathcal{J}$ such that*

a) *for each $J \in \mathcal{J}$, if $F \cap \operatorname{bd} J \neq \emptyset$, then $J \in \mathcal{J}'$,*

b) *for each $c \in F$, if c is a right-hand (left-hand) limit point of C, then c is a right-hand (respectively left-hand) limit point of the union $\bigcup \mathcal{J}'$,*

c) $\operatorname{cl} \bigcup \mathcal{J}' \subset F \cup \bigcup_{J \in \mathcal{J}'} \operatorname{cl} J.$

Remark 2.3. Let $f\colon \mathbb{R} \to \mathbb{R}$. Clearly, if the function f is Darboux upper semicontinuous, then $\overline{\lim}(f, x^-) = f(x) = \overline{\lim}(f, x^+)$ for each $x \in \mathbb{R}$.

Next two theorems characterize the maximal class with respect to maximums for the family of upper semicontinuous strong Świątkowski functions.

THEOREM 2.4. $\mathcal{M}_{\max}(\acute{\mathcal{S}}_{\mathrm{susc}}) \subset \acute{\mathcal{S}}_{\mathrm{susc}}$.

Proof. First assume that $f \notin \acute{\mathcal{S}}_s$. Then there are $\alpha < \beta$ and $y \in \mathrm{I}(f(\alpha), f(\beta))$ such that $f(x) \neq y$ for each $x \in (\alpha, \beta) \cap \mathscr{C}(f)$. Put $g = \min\{f(\alpha), f(\beta)\}$ and $h = \max\{f, g\}$. Then clearly $g \in \mathscr{C} \subset \acute{\mathcal{S}}_{\mathrm{susc}}$. Since $y \in \mathrm{I}(h(\alpha), h(\beta))$ and $h(x) \neq y$ for each $x \in (\alpha, \beta) \cap \mathscr{C}(h)$, we have $h \notin \acute{\mathcal{S}}_s$. So, $h \notin \acute{\mathcal{S}}_{\mathrm{susc}}$, whence $f \notin \mathcal{M}_{\max}(\acute{\mathcal{S}}_{\mathrm{susc}})$.

Now assume that $f \notin$ usc. Then e.g., $f(x_0) < \overline{\lim}(f, x_0^+)$ for some $x_0 \in \mathbb{R}$. (The other case is analogous.) Put $g = f(x_0)$ and $h = \max\{f, g\}$. Then clearly $g \in \mathscr{C} \subset \acute{\mathcal{S}}_{\mathrm{susc}}$, and since

$$h(x_0) = g(x_0) = f(x_0) < \overline{\lim}(f, x_0^+) = \overline{\lim}(h, x_0^+),$$

$h \notin$ usc. So, $h \notin \acute{\mathcal{S}}_{\mathrm{susc}}$, whence $f \notin \mathcal{M}_{\max}(\acute{\mathcal{S}}_{\mathrm{susc}})$. □

THEOREM 2.5. *Assume that $f \in \acute{\mathcal{S}}_{\mathrm{susc}}$. The following conditions are equivalent*:

a) *there is a function $g \in \acute{\mathcal{S}}_{\mathrm{susc}}$ such that $\max\{f, g\} \notin \acute{\mathcal{S}}_s$,*

b) *there are: real numbers $a < b$, a nowhere dense G_δ-set $A \subset (a, b)$, a point $x_0 \in A$, and a subfamily $\mathcal{J}$ of the family of all components of $[a, b] \setminus \mathrm{cl}\, A$ such that*
   - (i) $\mathrm{cl}\, A \subset \mathrm{cl} \bigcup \mathcal{J}$,
   - (ii) $\bigcup_{J \in \mathcal{J}} \mathrm{bd}\, J \cap (a, b) \subset A \cap \mathrm{cl}[f > f(x_0)]$,
   - (iii) $\bigcup \mathcal{J} \subset \mathrm{int}\big([f < f(x_0)] \cup \big([f = f(x_0)] \setminus \mathscr{C}(f)\big)\big)$,
   - (iv) $\underline{\lim}(f, \bigcup \mathcal{J}, x) < f(x_0)$ *for each* $x \in A$.

Proof. Assume that $f \in \acute{\mathcal{S}}_{\mathrm{susc}}$.

NECESSITY.

Let $g \in \acute{\mathcal{S}}_{\mathrm{susc}}$ and $h = \max\{f, g\} \notin \acute{\mathcal{S}}_s$. Then there are $a < b$ and $y \in \mathrm{I}(h(a), h(b))$ such that

$$h(x) \neq y \qquad \text{for each} \quad x \in (a, b) \cap \mathscr{C}(h). \tag{1}$$

Since the maximum of two upper semicontinuous functions is upper semicontinuous (see e.g. [2: p. 83]), $h \in$ usc.

Define

$$G_1 = \mathrm{int}[h \leq y] \quad \text{and} \quad G_2 = \mathrm{int}[h \geq y].$$

Clearly sets G_1 and G_2 are nonempty, open and disjoint in $[a, b]$. Assume that $\mathcal{I}_1$ and $\mathcal{I}_2$ are families of all components of $G_1 \cap (a, b)$ and $G_2 \cap (a, b)$, respectively.

Moreover let $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$ and $C = [a,b] \setminus (G_1 \cup G_2)$. Obviously the set C is closed and nonempty, and since $h \in \mathrm{usc}$,

$$C \cap (a,b) \subset [h \geq y]. \tag{2}$$

We will show that C is nowhere dense.

Suppose, on the contrary, that $\operatorname{int}\operatorname{cl} C \neq \emptyset$. Then there is an open interval $I \subset C$. Using condition (1) we obtain that $h(x) \neq y$ for some $x \in I$. If $h(x) > y$, then $f(x) > y$ or $g(x) > y$. Without loss of generality we can assume that $f(x) > y$. Since $f \in \acute{\mathcal{S}}_s$, there is a $t \in I \cap \mathscr{C}(f)$ such that $f(t) > y$. So, $(t-\delta, t+\delta) \cap I \subset [f > y]$ for some $\delta > 0$, whence $(t-\delta, t+\delta) \cap I \subset [h > y]$. It proves that $(t-\delta, t+\delta) \cap I \subset G_2$, an impossibility. If $h(x) < y$, then $f(x) < y$ and $g(x) < y$. Since $f, g \in \mathrm{usc}$, there is a $\delta > 0$ such that $(x-\delta, x+\delta) \cap I \subset [f < y] \cap [g < y]$. Hence $(x-\delta, x+\delta) \cap I \subset [h < y]$, which proves that $(x-\delta, x+\delta) \cap I \subset G_1$, a contradiction. So, the set C is nowhere dense.

Now we will show some properties of the set $\bigcup_{I \in \mathcal{I}_1} \operatorname{bd} I$. First observe that

$$\bigcup_{I \in \mathcal{I}_1} \operatorname{bd} I \cap (a,b) \subset \operatorname{cl}[h > y]. \tag{3}$$

Moreover,

$$\bigcup_{I \in \mathcal{I}_1} \operatorname{bd} I \cap (a,b) \subset \operatorname{cl}[f > y]. \tag{4}$$

Indeed, let $x \in (a,b)$ and $x = \sup I$ for some $I \in \mathcal{I}_1$. (The case $x = \inf I$ for some $I \in \mathcal{I}_1$ is analogous.) If there was a $\delta > 0$ such that $f(t) \leq y$ for each $t \in (x, x+\delta)$, then from (3) and since $h = \max\{f, g\}$, there was a $t_\delta \in (x, x+\delta)$ with $g(t_\delta) > y$. But $I \subset [h \leq y]$, whence, by (1), we would have $g(z) < y$ for some $z \in I$. Since $g \in \acute{\mathcal{S}}_s$, there was a $t_0 \in (z, t_\delta) \cap \mathscr{C}(g)$ such that $g(t_0) = y$. Moreover $(z, t_\delta) \subset [f \leq y]$. Using $h = \max\{f, g\}$ one more time, we would obtain $g(t_0) = h(t_0) = y$ and $t_0 \in \mathscr{C}(h)$, which contradicts (1).

In the same way we can prove that

$$\bigcup_{I \in \mathcal{I}_1} \operatorname{bd} I \cap (a,b) \subset \operatorname{cl}[g > y]. \tag{5}$$

Finally we will show that

$$h(x) = f(x) = g(x) = y \qquad \text{for each} \quad x \in \bigcup_{I \in \mathcal{I}_1} \operatorname{bd} I \cap (a,b). \tag{6}$$

Let $x \in (a,b)$ and $x = \sup I$ for some $I \in \mathcal{I}_1$. (The case $x = \inf I$ for some $I \in \mathcal{I}_1$ is analogous.) By condition (2), $h(x) \geq y$. If $h(x) > y$, then $f(x) > y$ or $g(x) > y$. But $I \subset [h \leq y]$, whence $I \subset [f \leq y] \cap [g \leq y]$. It contradicts $f, g \in \acute{\mathcal{S}}_s \subset \mathscr{D}$ and consequently $h(x) = y$. If $g(x) < y$, then since $g \in \mathrm{usc}$, there was a $\delta > 0$ such that $(x-\delta, x+\delta) \subset [g < y]$, which contradicts (5). Thus $y = h(x) \geq g(x) \geq y$, whence $g(x) = y$. In the similar way we can show that $f(x) = y$. So, condition (6) is fulfilled.

Next we claim that

$$I \subset \operatorname{int}\bigl([f<y] \cup \bigl([f=y] \setminus \mathscr{C}(f)\bigr)\bigr) \qquad \text{for each} \quad I \in \mathcal{I}_1. \tag{7}$$

Indeed, fix an $I \in \mathcal{I}_1$. Since $h = \max\{f,g\}$, we have $[h<y] \subset [f<y]$, and by (1),

$$[h=y] \cap (a,b) \subset [f<y] \cup \bigl([f=y] \setminus \mathscr{C}(f)\bigr).$$

Hence, using definition of $\mathcal{I}_1$, we obtain

$$I \subset \operatorname{int}[h \le y] \cap (a,b) \subset \operatorname{int}\bigl([f<y] \cup \bigl([f=y] \setminus \mathscr{C}(f)\bigr)\bigr),$$

as claimed.

Now we will prove that all our requirements are fulfilled. For each $n \in \mathbb{N}$ define

$$F_n = \operatorname{cl}\bigl(\bigl[h > y + \tfrac{1}{n}\bigr] \cap C\bigr).$$

Let $F = \bigcup_{n \in \mathbb{N}} F_n \cup \{a,b\}$. Then F is an F_σ-set. Define $A = C \setminus F$. Clearly A is a nowhere dense G_δ-set and $A \subset (a,b)$. Now we will show that

$$C \subset \operatorname{cl} \bigcup \mathcal{I}_1 \cup \{a,b\}. \tag{8}$$

Let $x \in C \setminus \{a,b\}$. If (8) was not fulfilled, then there was a $\delta > 0$ such that $(x-\delta, x+\delta) \subset [a,b]$ and $(x-\delta,x+\delta) \cap \bigcup \mathcal{I}_1 = \emptyset$. But then $(x-\delta,x+\delta) \subset C \cup \bigcup \mathcal{I}_2$, and by (2), we would have $(x-\delta,x+\delta) \subset [h \ge y]$. Hence $(x-\delta,x+\delta) \subset G_2$, a contradiction.

Moreover

$$\bigcup_{I \in \mathcal{I}_1} \operatorname{bd} I \subset A \cup \{a,b\}. \tag{9}$$

Indeed, let $x \in \operatorname{bd} I \setminus \{a,b\}$ for some $I \in \mathcal{I}_1$. Hence obviously $x \in C$. If $x \in F$, then $x \in F_n$ for some $n \in \mathbb{N}$. Since $h \in \mathrm{usc}$, we would have $h(x) \ge y + \frac{1}{n}$, which contradicts (6). Therefore $x \in A$, whence condition (9) holds.

Now observe that conditions (8) and (9) imply $\operatorname{cl} A \cup \{a,b\} = C \cup \{a,b\}$. So, we can assume that $\mathcal{J} = \mathcal{I}_1$. Then $\mathcal{J}$ is a subfamily of the family of all components of $[a,b] \setminus \operatorname{cl} A$. Choose an $I_0 \in \mathcal{J}$ such that $\sup I_0 \cap \{a,b\} \ne \emptyset$ and let $x_0 = \sup I_0$. Clearly $x_0 \in A$. It remains to prove that conditions (i)–(iv) are fulfilled.

Condition (i) follows from (8). (Recall that $A \subset C \setminus \{a,b\}$.) Using (9) we obtain that

$$\bigcup_{J \in \mathcal{J}} \operatorname{bd} J \cap (a,b) \subset A.$$

Since $x_0 \in \bigcup_{J \in \mathcal{J}} \operatorname{bd} J \cap (a,b)$, by (6), $f(x_0) = y$. Therefore, by (4),

$$\bigcup_{J \in \mathcal{J}} \operatorname{bd} J \cap (a,b) \subset \operatorname{cl}[f > f(x_0)].$$

So, condition (ii) is fulfilled. Condition (iii) holds directly from (7). Finally, fix an $x \in A$. Observe that, by (2), $h(x) \ge y$. But if $h(x) > y$, then $x \in F$, a contradiction. Hence $h(x) = y$. Taking into account that $h \in \text{usc}$, $f, g \in \acute{\mathcal{S}}_s$, and $h = \max\{f, g\}$, we conclude that $\overline{\lim}(h, x^-) = \overline{\lim}(h, x^+) = y$. Moreover $C \cup \bigcup \mathcal{I}_2 \subset [h \geq y]$. So, if $\underline{\lim}\big(h, \bigcup \mathcal{I}_1, x^-\big) = \underline{\lim}\big(h, \bigcup \mathcal{I}_1, x^+\big) = y$, then $x \in \mathscr{C}(h)$, which contradicts (1). Therefore $\underline{\lim}\big(h, \bigcup \mathcal{I}_1, x^-\big) < y$ or $\underline{\lim}\big(h, \bigcup \mathcal{I}_1, x^+\big) < y$, whence by (6),

$$\underline{\lim}(f, \bigcup \mathcal{J}, x) < y = f(x_0).$$

This completes first part of the proof.

Sufficiency.

Now assume that there are real numbers $a < b$, a nowhere dense G_δ-set $A \subset (a, b)$, a point $x_0 \in A$, and a subfamily $\mathcal{J}$ of the family of all components of $[a, b] \setminus \operatorname{cl} A$ such that conditions (i)–(iv) are fulfilled. First observe that using assumptions (ii), (iii), and the fact that $f \in \acute{\mathcal{S}}_{\text{susc}}$, we have

$$f(x) = f(x_0) \qquad \text{for each} \quad x \in \bigcup_{J \in \mathcal{J}} \operatorname{bd} J \cap (a, b). \tag{10}$$

Since $\operatorname{cl} A$ is nowhere dense we can write $\operatorname{cl} A$ as the disjoint union $\operatorname{cl} A = C \cup P$, where P is countable and C is perfect. We consider two cases.

Case I. $P \neq \emptyset$.

Then, by assumption (ii), there is an isolated in A point $z_0 \in P \cap (a, b) \cap \bigcup_{J \in \mathcal{J}} \operatorname{bd} J$. Let $z_0 = \sup J$ for some $J \in \mathcal{J}$. (If $z_0 = \inf J$ for some $J \in \mathcal{J}$ we proceed analogously.) Then, by (10), $f(z_0) = f(x_0)$. This fact and assumption (iv) imply that $z_0 \notin \mathscr{C}^-(f)$. Using assumption (iii) we obtain that

$$f(x) < f(x_0) \qquad \text{for each} \quad x \in J \cap \mathscr{C}(f). \tag{11}$$

Moreover, by assumption (ii) and since $f \in \acute{\mathcal{S}}_s$, there is a sequence $(x_n) \subset \mathscr{C}(f)$ such that $x_n \to z_0^+$ and $f(x_n) > f(x_0)$ for each $n \in \mathbb{N}$. Hence, for each $n \in \mathbb{N}$, there is a $\delta_n > 0$ such that $f(x) > f(x_0)$ for every $x \in (x_n - \delta_n, x_n + \delta_n)$. Without loss of generality we can assume that $x_{n+1} + \delta_{n+1} < x_n - \delta_n$ for each $n \in \mathbb{N}$. Define the function $g \colon \mathbb{R} \to \mathbb{R}$ as follows:

$$g(x) = \begin{cases} f(x) & \text{if } x \in (-\infty, z_0], \\ f(x_0) & \text{if } x \in \{x_n : \ n \in \mathbb{N}\} \cup [x_1, \infty), \\ f(x_{n+1}) & \text{if } x \in \bigcup_{n=1}^{\infty} [x_{n+1} + \delta_{n+1}, x_n - \delta_n], \\ \text{linear} & \text{in each interval } [x_{n+1}, x_{n+1} + \delta_{n+1}] \text{ and } [x_n - \delta_n, x_n], \\ & n \in \mathbb{N}. \end{cases}$$

Observe that $g \restriction (-\infty, z_0] = f \restriction (-\infty, z_0] \in \acute{\mathcal{S}}_{\text{susc}}$ and $g \restriction [z_0, \infty) \in \mathscr{C}$. So, clearly $g \in \text{usc}$. Moreover, $g(z_0) = g[(x_n)]$ and $(x_n) \subset \mathscr{C}(g)$. Thus, by Lemma 2.1, $g \in \acute{\mathcal{S}}_s$. Now we will show that $h = \max\{f, g\} \notin \acute{\mathcal{S}}_s$.

Take an $\alpha \in J \cap \mathscr{C}(f)$ and let $\beta = x_1 - \delta_1$. Notice that, by (11), for each $x \in [\alpha, z_0) \cap \mathscr{C}(f)$

$$h(x) = g(x) = f(x) < f(x_0).$$

Now fix an $x \in (z_0, \beta]$. Observe that $h(x) > f(x_0)$. Indeed,

- if $x \in (x_n - \delta_n, x_n + \delta_n)$ for some $n \in \mathbb{N}$, then $h(x) \geq f(x) > f(x_0)$, and
- if $x \in [x_{n+1} + \delta_{n+1}, x_n - \delta_n]$ for some $n \in \mathbb{N}$, then $h(x) \geq g(x) > f(x_0)$.

Hence in particular $f(x_0) \in (h(\alpha), h(\beta))$. Moreover, since $z_0 \notin \mathscr{C}^-(f)$ and $f = g$ on $(-\infty, z_0]$, we have $z_0 \notin \mathscr{C}(h)$. Therefore $h(x) \neq f(x_0)$ for each $x \in (\alpha, \beta) \cap \mathscr{C}(h)$. Consequently $h = \max\{f, g\} \notin \acute{\mathcal{S}}_s$.

Case II. $P = \emptyset$.

Then $C \neq \emptyset$ and $C = \operatorname{cl} A$. (Recall that C is perfect.) Define

$$c = \inf\{x \in [a,b] :\ x \in C\} \quad \text{and} \quad d = \sup\{x \in [a,b] :\ x \in C\}.$$

Observe that $c, d \in C$. Let $\mathcal{I}$ be the family of all components of $[c,d] \setminus C$. Define

$$\mathcal{I}' = \{I \in \mathcal{I} :\ I \cap [f > f(x_0)] \neq \emptyset\}.$$

By (10) and assumption (ii), $\mathcal{I}' \neq \emptyset$. Taking into account that the set C is perfect and using assumptions (i) and (ii), we obtain that

$$C = \operatorname{cl} A \subset \operatorname{cl} \bigcup_{J \in \mathcal{J}} \operatorname{bd} J \subset \operatorname{cl}[f > f(x_0)].$$

Since $f \in \acute{\mathcal{S}}_{\mathrm{susc}}$, we have $C \subset \operatorname{cl} \bigcup \mathcal{I}'$. Now define

$$A_1 = A \cap C \setminus \bigcup_{I \in \mathcal{I}'} \operatorname{bd} I. \tag{12}$$

Since A is a G_δ-set, A_1 is a G_δ-set, too. Then $C \setminus A_1$ is an F_σ-set, whence there is a sequence (F_n) consisting of closed sets such that

$$C \setminus A_1 = \bigcup_{n \in \mathbb{N}} F_n. \tag{13}$$

Define $F'_0 = \emptyset$. For each $n \in \mathbb{N}$, use Lemma 2.2 to construct a sequence of sets (F'_n) and a sequence of families of intervals $(\mathcal{I}'_n)$ such that

$$\mathcal{I}'_n \subset \mathcal{I}', \tag{14}$$

$$F'_n = F_n \cup \bigcup_{k<n} \big(F'_k \cup \bigcup_{I \in \mathcal{I}'_k} \operatorname{bd} I\big) \tag{15}$$

$$\text{for each } I \in \mathcal{I}', \text{ if } F'_n \cap \operatorname{bd} I \neq \emptyset, \text{ then } I \in \mathcal{I}'_n, \tag{16}$$

$$\begin{gathered}\text{for each } c \in F'_n, \text{ if } c \text{ is a right-hand (left-hand) limit point of } C,\\ \text{then } c \text{ is a right-hand (left-hand) limit point of the union } \bigcup \mathcal{I}'_n,\end{gathered} \tag{17}$$

$$\operatorname{cl} \bigcup \mathcal{I}'_n \subset F'_n \cup \bigcup_{I \in \mathcal{I}'_n} \operatorname{cl} I. \tag{18}$$

Note that, by (18), for each $k < n$, the set $F'_k \cup \{\operatorname{bd} I : \ I \in \mathcal{I}'_k\}$ is closed. So, by (15), the set F'_n is also closed and $F'_n \subset C \setminus A_1$. Moreover, by (16), $\bigcup\limits_{n\in\mathbb{N}} \mathcal{I}'_n = \mathcal{I}'$.

Put

$$n_I = \min\{n \in \mathbb{N} : \ I \in \mathcal{I}'_n\}, \quad N_x = \min\{n \in \mathbb{N} : \ x \in F'_n\},$$

and

$$n_x = \begin{cases} N_x - 1 & \text{if } x \in \{\operatorname{bd} I : \ I \in \mathcal{I}'\} \text{ and } x \text{ is a right-hand (left-hand)} \\ & \text{limit point of the union } \bigcup \mathcal{I}'_{N_x - 1}, \\ N_x & \text{otherwise.} \end{cases}$$

Fix an $I = (a_I, b_I) \in \mathcal{I}'$. Observe that, if $x \in \operatorname{bd} I$, then by (15), $\frac{1}{n_I+1} \leq \frac{1}{n_x} \leq \frac{1}{n_I}$. Moreover, since $f \in \acute{\mathcal{S}}_s$ and $I \cap [f > f(x_0)] \neq \emptyset$, there is a $z \in I \cap \mathscr{C}(f)$ with $f(x) > f(x_0)$. So, there is a $\delta > 0$ such that $[z-\delta, z+\delta] \subset I$ and $f(x) > f(x_0)$ for each $x \in (z - \delta, z + \delta)$. Define the function $g_I \colon \operatorname{cl} I \to \mathbb{R}$ as follows:

$$g_I(x) = \begin{cases} f(x_0) & \text{if } x = z, \\ f(x_0) + \frac{1}{n_I} & \text{if } x \in \{z - \delta, z + \delta\}, \\ f(x_0) + \frac{1}{n_x} & \text{if } x \in \operatorname{bd} I, \\ \text{linear} & \text{in intervals } [a_I, z - \delta],\ [z - \delta, z],\ [z, z + \delta], \text{ and} \\ & [z + \delta, b_I]. \end{cases}$$

Further assume that $\mathcal{I}_1 = \mathcal{I} \setminus (\mathcal{I}' \cup \mathcal{J})$ and fix an $I = (a_I, b_I) \in \mathcal{I}_1$. Define the function $\varphi_I \colon \operatorname{cl} I \to \mathbb{R}$ as follows:

$$\varphi_I(x) = \begin{cases} f(x_0) & \text{if } x \in \operatorname{bd} I \setminus \bigcup\limits_{n\in\mathbb{N}} F'_n, \\ f(x_0) + \frac{1}{n_x} & \text{if } x \in \operatorname{bd} I \cap \bigcup\limits_{n\in\mathbb{N}} F'_n, \\ f(x_0) + |I| & \text{if } x = \frac{a_I + b_I}{2}, \\ \text{linear} & \text{in intervals } \left[a_I, \frac{a_I+b_I}{2}\right] \text{ and } \left[\frac{a_I+b_I}{2}, b_I\right]. \end{cases}$$

Now define the function $\psi \colon [c, d] \to \mathbb{R}$ by the formula:

$$\psi(x) = \begin{cases} f(x) & \text{if } x \in \operatorname{cl} I,\ I \in \mathcal{J}, \\ g_I(x) & \text{if } x \in \operatorname{cl} I,\ I \in \mathcal{I}', \\ \varphi_I(x) & \text{if } x \in \operatorname{cl} I,\ I \in \mathcal{I}_1, \\ f(x_0) + \frac{1}{n_x} & \text{if } x \in \bigcup\limits_{n\in\mathbb{N}} F'_n \setminus \bigcup\limits_{I\in\mathcal{I}} \operatorname{cl} I, \\ f(x_0) & \text{if } x \in A_1 \setminus \bigcup\limits_{I\in\mathcal{I}} \operatorname{cl} I. \end{cases}$$

Observe that $A_1 \subset [\psi = f(x_0)]$ and

$$[\psi < f(x_0)] \subset \bigcup \mathcal{J} \subset [\psi \le f(x_0)]. \tag{19}$$

We we will show that $\psi \in \acute{\mathcal{S}}_s$ usc. First we will prove that ψ is upper semicontinuous. Clearly $\psi \restriction \bigcup \mathcal{I} \in$ usc. So, let $x \in C$.

If $x \in A_1$, then $\psi(x) = f(x_0)$. Suppose that e.g., $\overline{\lim}(\psi, x^-) > f(x_0)$. (The other case is similar.) Without loss of generality we can assume that $x \neq \sup I$ for each $I \in \mathcal{I}$. Choose an $n_0 \in \mathbb{N}$ such that $\overline{\lim}(\psi, x^-) > f(x_0) + \frac{1}{n_0 - 1}$. By (19) and construction of ψ we obtain that

$$x \in \operatorname{cl}\Big(F'_{n_0} \cup \bigcup \mathcal{I}'_{n_0} \cup \bigcup_{I \in \mathcal{I}_1,\, |I| \ge \frac{1}{n_0}} I\Big) \cap (-\infty, x).$$

Since $A_1 \subset C$ and C is perfect, $x \notin \operatorname{cl} \bigcup_{I \in \mathcal{I}_1, |I| \ge \frac{1}{n_0}} I \cap (-\infty, x)$. Moreover, by (18), (15), (12), and (13),

$$\begin{aligned} A_1 \cap \operatorname{cl}(F'_{n_0} \cup \bigcup \mathcal{I}'_{n_0}) &\subset (A_1 \cap F'_{n_0}) \cup \Big(A_1 \cap \bigcup_{I \in \mathcal{I}'_{n_0}} \operatorname{cl} I\Big) \\ &\subset \Big(A_1 \cap \bigcup_{n \le n_0} F'_n\Big) \cup \Big(\Big(C \setminus \bigcup_{I \in \mathcal{I}'} \operatorname{bd} I\Big) \cap \bigcup_{I \in \mathcal{I}'} \operatorname{cl} I\Big) = \emptyset, \end{aligned}$$

a contradiction. So, ψ is upper semicontinuous on A_1.

If $x \notin A_1$, then $x \in \bigcup_{n \in \mathbb{N}} F_n$. Hence $x \in F'_n \setminus F'_{n-1}$ for some $n \in \mathbb{N}$. Suppose that e.g., $\overline{\lim}(\psi, x^-) > f(x_0) + \frac{1}{n}$. (The other case is similar.) If $x = \sup I$ for some $I \in \mathcal{I}$, then by (10) and construction of ψ we have $\overline{\lim}(\psi, x^-) \le f(x_0) + \frac{1}{n}$, a contradiction. So let $x \neq \sup I$ for each $I \in \mathcal{I}$. Note that $x \in \operatorname{cl}\big([c, x] \cap [\psi > f(x_0) + \frac{1}{n}]\big)$, whence $x \in \operatorname{cl} \bigcup \mathcal{I}'_{n-1}$. But by (18), $x \in \bigcup_{I \in \mathcal{I}'_{n-1}} \operatorname{cl} I$. Hence there is an $I \in \mathcal{I}$ such that $x = \sup I$, which is impossible. So, $\overline{\lim}(\psi, x^-) \le f(x_0) + \frac{1}{n}$ $= \psi(x)$. It follows that $\psi \in$ usc.

Now we will prove that for each $n \in \mathbb{N}$ and $\delta > 0$, if $x \neq c$ and $x \in F'_n \setminus \{\sup I : I \in \mathcal{I}\}$, then

$$\psi\big[(x - \delta, x) \cap \mathscr{C}(\psi)\big] \supset \big[f(x_0), f(x_0) + \tfrac{1}{n}\big]. \tag{20}$$

Let $n \in \mathbb{N}$, $\delta > 0$, $x \neq c$, and $x \in F'_n \setminus \{\sup I : I \in \mathcal{I}\}$. Then $x \in F'_n \cap \operatorname{cl}((-\infty, x) \cap C)$ and by (17), there is an $I \in \mathcal{I}'_n$ with $I \subset (x - \delta, x)$. Notice that $n_I \le n$. So,

$$\psi\big[(x - \delta, x) \cap \mathscr{C}(\psi)\big] \supset \psi\big[I \cap \mathscr{C}(\psi)\big] \supset \big[f(x_0), f(x_0) + \tfrac{1}{n}\big].$$

Similarly we can prove that for each $n \in \mathbb{N}$ and $\delta > 0$, if $x \neq d$ and $x \in F'_n \setminus \{\inf I : I \in \mathcal{I}\}$, then

$$\psi\big[(x, x + \delta) \cap \mathscr{C}(\psi)\big] \supset \big[f(x_0), f(x_0) + \tfrac{1}{n}\big].$$

Now we will show that $\psi \in \acute{\mathcal{S}}_s$. Let $\alpha, \beta \in [c,d]$, $\alpha < \beta$, and $y \in \mathrm{I}(\psi(\alpha), \psi(\beta))$. Assume that $\psi(\alpha) < \psi(\beta)$. (The other case is similar.) If $\alpha, \beta \in \operatorname{cl} I$ for some $I \in \mathcal{I}$, then since $\psi \restriction \operatorname{cl} I \in \acute{\mathcal{S}}_s$, there is a $t_0 \in (\alpha, \beta) \cap \mathscr{C}(\psi)$ with $\psi(t_0) = y$. So, assume that the opposite case holds. We consider two cases.

Case 1. If $y \geq f(x_0)$, then $\psi(\beta) > f(x_0)$ and $\beta \notin A_1$.

First assume that $\beta \notin \bigcup_{n \in \mathbb{N}} F'_n$ or $\beta \in \{\sup I : I \in \mathcal{I}\}$. Then there is an $I \in \mathcal{I}$ such that $\beta \in \operatorname{cl} I$ and $\alpha \notin \operatorname{cl} I$. If $y \in \mathrm{I}(\psi(\inf I), \psi(\beta))$, then since $\psi \restriction \operatorname{cl} I \in \acute{\mathcal{S}}_s$, there is a $t_0 \in (\inf I, \beta) \cap \mathscr{C}(\psi) \subset (\alpha, \beta) \cap \mathscr{C}(\psi)$ with $\psi(t_0) = y$. So, let $y \in [f(x_0), \psi(\inf I)]$.

- If $\inf I \in A_1$, then $\psi(\inf I) = f(x_0) = y$ and since $\inf I \in C \subset \operatorname{cl} \bigcup \mathcal{I}'$, there is an $I' \in \mathcal{I}'$ such that $I' \subset (\alpha, \inf I)$. Hence $\psi(t_0) = f(x_0) = y$ for some $t_0 \in I' \cap \mathscr{C}(\psi) \subset (\alpha, \beta) \cap \mathscr{C}(\psi)$.
- If $\inf I \in \bigcup_{n \in \mathbb{N}} F'_n$, then $\inf I \in F'_n \setminus F'_{n-1}$ for some $n \in \mathbb{N}$. By (20),

$$y \in \big[f(x_0), \psi(\inf I)\big] = \big[f(x_0), f(x_0) + \tfrac{1}{n}\big] \subset \psi\big[(\alpha, \inf I) \cap \mathscr{C}(\psi)\big].$$

So, there is a $t_0 \in (\alpha, \inf I) \cap \mathscr{C}(\psi) \subset (\alpha, \beta) \cap \mathscr{C}(\psi)$ with $\psi(t_0) = y$.

Now assume that $\beta \in \bigcup_{n \in \mathbb{N}} F'_n \setminus \{\sup I : I \in \mathcal{I}\}$. Then $\beta \in F'_n \setminus F'_{n-1}$ for some $n \in \mathbb{N}$. By (20),

$$y \in \big[f(x_0), \psi(\beta)\big) = \big[f(x_0), f(x_0) + \tfrac{1}{n}\big) \subset \psi\big[(\alpha, \beta) \cap \mathscr{C}(\psi)\big].$$

Consequently, there is a $t_0 \in (\alpha, \beta) \cap \mathscr{C}(\psi)$ with $\psi(t_0) = y$.

Case 2. If $y < f(x_0)$, then $\psi(\alpha) < f(x_0)$ and $\alpha \notin A_1$.

Then there is a $J \in \mathcal{J}$ such that $\alpha \in J$ and $\beta \notin \operatorname{cl} J$. Since, by (10), $\psi(\sup J) = f(x_0)$ and $\psi \restriction \operatorname{cl} J = f \restriction \operatorname{cl} J \in \acute{\mathcal{S}}_s$, there is a $t_0 \in (\alpha, \sup J) \cap \mathscr{C}(\psi) \subset (\alpha, \beta) \cap \mathscr{C}(\psi)$ with $\psi(t_0) = y$. It follows that $\psi \in \acute{\mathcal{S}}_s$.

Now define the function $g \colon \mathbb{R} \to \mathbb{R}$ as follows:

$$g(x) = \begin{cases} \psi(x) & \text{if } x \in [c,d], \\ \psi(c) & \text{if } x \in (-\infty, c], \\ \psi(d) & \text{if } x \in [d, \infty). \end{cases}$$

Then clearly $g \in \mathrm{usc}$ and by Lema 2.1, $g \in \acute{\mathcal{S}}_s$. Finally we must show that $h = \max\{f, g\} \notin \acute{\mathcal{S}}_s$. Take $\alpha, \beta \in [c,d]$ such that $\alpha \in \bigcup \mathcal{J}$, $\beta \in \bigcup \mathcal{I}'$, $\alpha < \beta$, and $h(\alpha) < f(x_0) < h(\beta)$. Obviously such numbers exist. It is easy to see that $[h = f(x_0)] \cap (\alpha, \beta) \subset \bigcup \mathcal{J} \cup A_1$. If $x \in J \cap [h = f(x_0)] \cap (\alpha, \beta)$ for some $J \in \mathcal{J}$, then since $f = g = h$ on J, using assumption (iii), we obtain that $x \notin \mathscr{C}(h)$. If $x \in A_1 \cap (\alpha, \beta)$, then since $A_1 \subset A$, by assumption (iv),

$$\underline{\lim}\big(f, \bigcup \mathcal{J}, x^+\big) = \underline{\lim}\big(h, \bigcup \mathcal{J}, x^+\big) < f(x_0) \leq h(x)$$

or

$$\underline{\lim}\big(f, \bigcup \mathcal{J}, x^-\big) = \underline{\lim}\big(h, \bigcup \mathcal{J}, x^-\big) < f(x_0) \leq h(x),$$

whence we also obtain that $x \notin \mathscr{C}(h)$. Consequently, $h(x) \neq f(x_0)$ for each $x \in (\alpha, \beta) \cap \mathscr{C}(h)$. So, $h = \max\{f, g\} \notin \acute{\mathcal{S}}_s$, which completes the proof. □

An immediate consequence of Theorem 2.5 is the following corollary.

Corollary 2.6. *$\mathscr{C} \subset \mathcal{M}_{\max}(\acute{\mathcal{S}}_{\mathrm{susc}})$.*

Proof. Suppose that $f \in \mathscr{C}$ and $f \notin \mathcal{M}_{\max}(\acute{\mathcal{S}}_{\mathrm{susc}})$. Then $f \in \acute{\mathcal{S}}_{\mathrm{susc}}$ and there is a function $g \in \acute{\mathcal{S}}_{\mathrm{susc}}$ such that $\max\{f, g\} \notin \acute{\mathcal{S}}_{\mathrm{susc}}$. Note that $\max\{f, g\} \in \mathrm{usc}$, whence $\max\{f, g\} \notin \acute{\mathcal{S}}_s$. Using condition (iv) of Theorem 2.5 we directly obtain that $f \notin \mathscr{C}$, a contradiction. □

Finally we will show that inclusions from Theorem 2.4 and Corollary 2.6 are proper.

***Example* 2.7.** There is a function $f \in \mathcal{M}_{\max}(\acute{\mathcal{S}}_{\mathrm{susc}})$ which is not continuous.

Construction. Define the function $f \colon \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} x & \text{if } x \in (-\infty, 0], \\ \sin \frac{1}{x} - 1 - x & \text{if } x \in (0, \infty). \end{cases}$$

Clearly f is upper semicontinuous but not continuous. Note that $\mathbb{R} \setminus \mathscr{C}(f) = \{0\}$. So, if $x \in \mathscr{C}(f)$, then condition (iv) of Theorem 2.5 is not fulfilled, and if $x = 0$, then condition (ii) of Theorem 2.5 is not satisfied. Hence using Theorem 2.5 we obtain that $\max\{f, g\} \in \acute{\mathcal{S}}_s$ for each function $g \in \acute{\mathcal{S}}_{\mathrm{susc}}$. Since the maximum of two upper semicontinuous functions is upper semicontinuous, we have $\max\{f, g\} \in \acute{\mathcal{S}}_{\mathrm{susc}}$. It proves that $f \in \mathcal{M}_{\max}(\acute{\mathcal{S}}_{\mathrm{susc}})$. □

Remark 2.8. There is an upper semicontinuous strong Świątkowski function f such that $f \notin \mathcal{M}_{\max}(\acute{\mathcal{S}}_{\mathrm{susc}})$.

Proof. By [7: Example 4.2] there are functions $f, g \in \acute{\mathcal{S}}_{\mathrm{susc}}$ with $\max\{f, g\} \notin \acute{\mathcal{S}}_s$, whence $f \notin \mathcal{M}_{\max}(\acute{\mathcal{S}}_{\mathrm{susc}})$. □

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