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MAXIMAL CLASSES FOR LOWER AND UPPER SEMICONTINUOUS STRONG ŚWIĄTKOWSKI FUNCTIONS

Abstract. In this paper, we characterize the maximal additive and multiplicative classes for lower and upper semicontinuous strong Świątkowski functions and lower and upper semicontinuous extra strong Świątkowski functions. Moreover, we characterize the maximal class with respect to maximums for lower semicontinuous strong Świątkowski functions and lower and upper semicontinuous extra strong Świątkowski functions.

1. Introduction

We use mostly standard terminology and notation. The letters $\mathbb{R}$ and $\mathbb{N}$ denote the real line and the set of positive integers, respectively. The symbols $\mathrm{I}(a,b)$ and $\mathrm{I}[a,b]$ denote the open and the closed interval with endpoints a and b, respectively. For each $A \subset \mathbb{R}$, we use the symbol χ_A to denote the characteristic function of A.

Let $f\colon I \to \mathbb{R}$, where I is a nondegenerate interval. The symbols $\mathcal{C}(f)$, $\mathcal{C}^+(f)$, $\mathcal{C}^-(f)$, and $\mathcal{A}(f)$ will stand for the set of points of continuity, right-hand continuity, left-hand continuity of f, and the set of all local maximums (not necessarily strict) of f, respectively. We say that f is a *Darboux function* ($f \in \mathcal{D}$), if it maps connected sets onto connected sets. We say that f is a *strong Świątkowski function* [3] ($f \in \acute{\mathcal{S}}_s$), if whenever $\alpha, \beta \in I$, $\alpha < \beta$, and $y \in \mathrm{I}(f(\alpha), f(\beta))$, there is an $x_0 \in (\alpha,\beta) \cap \mathcal{C}(f)$ such that $f(x_0) = y$. We say that f is an *extra strong Świątkowski function* [7] ($f \in \acute{\mathcal{S}}_{es}$), if whenever $\alpha, \beta \in I$, $\alpha \neq \beta$, and $y \in \mathrm{I}[f(\alpha), f(\beta)]$, there is an $x_0 \in \mathrm{I}[\alpha,\beta] \cap \mathcal{C}(f)$ such that $f(x_0) = y$.

Observe that $\acute{\mathcal{S}}_{es} \subset \acute{\mathcal{S}}_s \subset \mathcal{D}$. To prove that the first inclusion is proper consider the function $x \mapsto \sin x^{-1} + x + 1$ for $x > 0$ and $x \mapsto 0$ for $x \leq 0$. It

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is easy to see that such function belongs to the class $\acute{\mathcal{S}}_s \backslash \acute{\mathcal{S}}_{es}$. Moreover, the function $x \mapsto \sin x^{-1} + x + 1$ for $x > 0$ and $x \mapsto x$ for $x \le 0$ is Darboux and it has not the strong Świątkowski property, which proves that the second inclusion is proper, too.

The symbols $\mathcal{C}$, lsc, and usc denote families of all continuous and lower and upper semicontinuous functions, respectively. If $\mathcal{L}$ and $\mathcal{F}$ are families of real functions, then we will write $\mathcal{L}\mathcal{F}$ instead of $\mathcal{L} \cap \mathcal{F}$. We say that $f \in \mathcal{C}onst$ if and only if $f[I]$ is a singleton. Finally,

$$\mathcal{C}onst_+ = \{f \in \mathcal{C}onst : \ f > 0 \ \text{ on } I\}.$$

Moreover, for each $x \in I$ we write

$$\overline{\lim}(f, x^+) = \overline{\lim_{t \to x^+}} \ f(t).$$

Similarly, we define the symbols $\overline{\lim}(f, x^-)$, $\underline{\lim}(f, x^+)$, and $\underline{\lim}(f, x^-)$.

If $\mathcal{L}$ and $\mathcal{L}_1$ are families of real functions, then we define:

$$\begin{aligned}
\mathcal{M}_a(\mathcal{L}_1, \mathcal{L}) &= \big\{f : \ (\forall g \in \mathcal{L}_1) \ f + g \in \mathcal{L}\big\}, \\
\mathcal{M}_m(\mathcal{L}_1, \mathcal{L}) &= \big\{f : \ (\forall g \in \mathcal{L}_1) \ fg \in \mathcal{L}\big\}, \\
\mathcal{M}_{\max}(\mathcal{L}_1, \mathcal{L}) &= \big\{f : \ (\forall g \in \mathcal{L}_1) \ \max\{f, g\} \in \mathcal{L}\big\}.
\end{aligned}$$

Moreover we let

$$\mathcal{M}_a(\mathcal{L}) = \mathcal{M}_a(\mathcal{L}, \mathcal{L}), \quad \mathcal{M}_m(\mathcal{L}) = \mathcal{M}_m(\mathcal{L}, \mathcal{L}), \quad \mathcal{M}_{\max}(\mathcal{L}) = \mathcal{M}_{\max}(\mathcal{L}, \mathcal{L}).$$

The above classes are called the maximal additive class for $\mathcal{L}$, the maximal multiplicative class for $\mathcal{L}$, and the maximal class with respect to maximums for $\mathcal{L}$, respectively.

Remark 1.1. Clearly if $\mathcal{L}' \subset \mathcal{L}$ and $\mathcal{L}_1' \supset \mathcal{L}_1$, then $\mathcal{M}_a(\mathcal{L}_1', \mathcal{L}') \subset \mathcal{M}_a(\mathcal{L}_1, \mathcal{L})$. Similar inclusions hold for $\mathcal{M}_m$ and $\mathcal{M}_{\max}$.

In 2003, I proved that $\mathcal{M}_a(\acute{\mathcal{S}}_s) = \mathcal{M}_m(\acute{\mathcal{S}}_s) = \mathcal{M}_{\max}(\acute{\mathcal{S}}_s) = \mathcal{M}_a(\acute{\mathcal{S}}_{es}) = \mathcal{M}_m(\acute{\mathcal{S}}_{es}) = \mathcal{M}_{\max}(\acute{\mathcal{S}}_{es}) = \mathcal{C}onst$ [5, Corollaries 3.2, 3.4, and 3.6]. Recently, I characterized the maximal class with respect to maximums for upper semicontinuous strong Świątkowski functions. It turns out that $\mathcal{M}_{\max}(\acute{\mathcal{S}}_s usc)$ consists of upper semicontinuous strong Świątkowski functions which fulfilled some special conditions [8, Theorem 2.5]. In this paper, we characterize the maximal additive and multiplicative classes for families $\acute{\mathcal{S}}_s usc$, $\acute{\mathcal{S}}_s lsc$, $\acute{\mathcal{S}}_{es} usc$ and $\acute{\mathcal{S}}_{es} lsc$ (Theorems 3.1 and 3.2) and the maximal class with respect to maximums for families $\acute{\mathcal{S}}_s lsc$, $\acute{\mathcal{S}}_{es} lsc$ and $\acute{\mathcal{S}}_{es} usc$ (Theorems 3.3 and 3.4).

2. Auxiliary lemmas

Lemma 2.1 is due to A. Maliszewski [4, Lemma 1].

Lemma 2.1. *Let $M \in \mathbb{R}$ and assume that a function $g\colon [a,b] \to (-\infty, M)$ is upper semicontinuous both at a and at b. Then there is a continuous function $\psi\colon [a,b] \to [\min\{g(a), g(b)\}, M]$ such that $\psi = g$ on $\{a,b\}$ and $\psi > g$ on (a,b).*

The proof of Lemma 2.2 we can find in [7, Theorem 3.1].

Lemma 2.2. *For each function $f\colon \mathbb{R} \to \mathbb{R}$, the following conditions are equivalent:*

a) $f \in \acute{\mathcal{S}}_{es}$,
b) $f \in \mathcal{D}$ *and* $f[I] = f[I \cap \mathcal{C}(f)]$, *for each nondegenerate interval* I,
c) $f \in \mathcal{D}$ *and* $f(x) \in f[\mathrm{I}[x,t] \cap \mathcal{C}(f)]$, *for each* $x \in \mathbb{R}$ *and each* $t \in \mathbb{R}\setminus\{x\}$.

The next lemma is probably known, but I could not find an appropriate reference and prove it in [5, Lemma 2.4].

Lemma 2.3. *If $f\colon \mathbb{R} \to \mathbb{R}$ then the set $f[\mathcal{A}(f)]$ is at most countable.*

The proof of Lemma 2.4 we can find in [6, Lemma 3.3].

Lemma 2.4. *Assume that $I \subset \mathbb{R}$ is an interval, $g\colon I \to \mathbb{R}$, and $h\colon \mathbb{R} \to \mathbb{R}$. If $g, h \in \acute{\mathcal{S}}_s$ then $h \circ g \in \acute{\mathcal{S}}_s$.*

Now we will prove an analogous lemma for the family $\acute{\mathcal{S}}_{es}$.

Lemma 2.5. *Assume that $I \subset \mathbb{R}$ is an interval, $g\colon I \to \mathbb{R}$, and $h\colon \mathbb{R} \to \mathbb{R}$. If $g, h \in \acute{\mathcal{S}}_{es}$ then $h \circ g \in \acute{\mathcal{S}}_{es}$.*

Proof. Let $x \in I$ and $t \in I\setminus\{x\}$. If $g\restriction \mathrm{I}[x,t] \in \mathcal{C}onst$, then $(h \circ g)\restriction \mathrm{I}[x,t] \in \mathcal{C}onst$ and

$$(h \circ g)(x) \in (h \circ g)[\mathrm{I}[x,t] \cap \mathcal{C}(h \circ g)].$$

In the other case, since $g \in \acute{\mathcal{S}}_{es} \subset \mathcal{D}$ then $g[\mathrm{I}[x,t]]$ is a nondegenerate interval. Since $h \in \acute{\mathcal{S}}_{es}$, by Lemma 2.2 we have

$$\begin{aligned}(h \circ g)(x) \in h\big[g[\mathrm{I}[x,t]]\big] &= h\big[g[\mathrm{I}[x,t]] \cap \mathcal{C}(h)\big] = h\big[g[\mathrm{I}[x,t] \cap \mathcal{C}(g)] \cap \mathcal{C}(h)\big] \\ &\subset h\big[g[\mathrm{I}[x,t] \cap \mathcal{C}(h \circ g)]\big] = (h \circ g)[\mathrm{I}[x,t] \cap \mathcal{C}(h \circ g)].\end{aligned}$$

Clearly $h \circ g \in \mathcal{D}$. By Lemma 2.2, we obtain that $h \circ g \in \acute{\mathcal{S}}_{es}$. ∎

Lemma 2.6 and Remark 2.7 are evident.

Lemma 2.6. *Assume that $I \subset \mathbb{R}$ is an interval, $g\colon I \to \mathbb{R}$, and $h\colon \mathbb{R} \to \mathbb{R}$. If $g \in lsc$ and h is continuous and increasing then $h \circ g \in lsc$.*

Remark 2.7. Let $f\colon \mathbb{R} \to \mathbb{R}$. If $f \in \acute{\mathcal{S}}_s lsc$ then $\underline{\underline{\lim}}(f, x^-) = f(x) = \underline{\underline{\lim}}(f, x^+)$, for each $x \in \mathbb{R}$, and if $f \in \acute{\mathcal{S}}_s usc$ then $\overline{\overline{\lim}}(f, x^-) = f(x) = \overline{\overline{\lim}}(f, x^+)$, for each $x \in \mathbb{R}$.

3. Main result

Theorem 3.1. $\mathcal{M}_a(\acute{\mathcal{S}}_s lsc) = \mathcal{M}_a(\acute{\mathcal{S}}_{es} lsc) = \mathcal{M}_a(\acute{\mathcal{S}}_s usc) = \mathcal{M}_a(\acute{\mathcal{S}}_{es} usc) = \mathcal{C}onst$.

Proof. The proof of the inclusion $\mathcal{M}_a(\acute{\mathcal{S}}_{es} lsc, \acute{\mathcal{S}}_s lsc) \subset \mathcal{C}onst$ is similar to the proof of [5, Theorem 3.1]. So, we will only prove the inclusion $\mathcal{M}_a(\acute{\mathcal{S}}_{es} usc, \acute{\mathcal{S}}_s usc) \subset \mathcal{C}onst$. Let $f \notin \mathcal{C}onst$. It follows that $-f \notin \mathcal{C}onst$ and since $\mathcal{M}_a(\acute{\mathcal{S}}_{es} lsc, \acute{\mathcal{S}}_s lsc) \subset \mathcal{C}onst$, the function $-f \notin \mathcal{M}_a(\acute{\mathcal{S}}_{es} lsc, \acute{\mathcal{S}}_s lsc)$. Hence, there is a function $\overline{g} \in \acute{\mathcal{S}}_{es} lsc$ such that $-f + \overline{g} \notin \acute{\mathcal{S}}_s lsc$. Put $g = -\overline{g}$. Then $g \in \acute{\mathcal{S}}_{es} usc$ and $-(f+g) = -f + \overline{g} \notin \acute{\mathcal{S}}_s lsc$. Thus $f + g \notin \acute{\mathcal{S}}_s usc$, whence $f \notin \mathcal{M}_a(\acute{\mathcal{S}}_{es} usc, \acute{\mathcal{S}}_s usc)$. Since

$$\mathcal{M}_a(\acute{\mathcal{S}}_{es} lsc, \acute{\mathcal{S}}_s lsc) \subset \mathcal{C}onst \subset \mathcal{M}_a(\acute{\mathcal{S}}_s lsc) \cap \mathcal{M}_a(\acute{\mathcal{S}}_{es} lsc)$$

and

$$\mathcal{M}_a(\acute{\mathcal{S}}_{es} usc, \acute{\mathcal{S}}_s usc) \subset \mathcal{C}onst \subset \mathcal{M}_a(\acute{\mathcal{S}}_s usc) \cap \mathcal{M}_a(\acute{\mathcal{S}}_{es} usc),$$

using Remark 1.1, we obtain that

$$\mathcal{M}_a(\acute{\mathcal{S}}_s lsc) = \mathcal{M}_a(\acute{\mathcal{S}}_{es} lsc) = \mathcal{M}_a(\acute{\mathcal{S}}_s usc) = \mathcal{M}_a(\acute{\mathcal{S}}_{es} usc) = \mathcal{C}onst. \ \blacksquare$$

Theorem 3.2. $\mathcal{M}_m(\acute{\mathcal{S}}_s lsc) = \mathcal{M}_m(\acute{\mathcal{S}}_{es} lsc) = \mathcal{M}_m(\acute{\mathcal{S}}_s usc) = \mathcal{M}_m(\acute{\mathcal{S}}_{es} usc) = \mathcal{C}onst_+ \cup \{\chi_\varnothing\}$.

Proof. First, we will show that

$$\mathcal{M}_m(\acute{\mathcal{S}}_{es} lsc, \acute{\mathcal{S}}_s lsc) \subset \mathcal{C}onst_+ \cup \{\chi_\varnothing\}. \tag{1}$$

Let the function $f \notin \mathcal{C}onst_+ \cup \{\chi_\varnothing\}$. If $f \notin \acute{\mathcal{S}}_s lsc$ then $\chi_\mathbb{R} \in \acute{\mathcal{S}}_{es} lsc$ and $f = f \cdot \chi_\mathbb{R} \notin \acute{\mathcal{S}}_s lsc$, whence $f \notin \mathcal{M}_m(\acute{\mathcal{S}}_{es} lsc, \acute{\mathcal{S}}_s lsc)$. So, we can assume that $f \in \acute{\mathcal{S}}_s lsc$.

If $f \in \acute{\mathcal{S}}_s lsc \setminus \mathcal{C}$ then by Remark 2.7, $f(x_0) < \overline{\lim}(f, x_0^+)$ or $f(x_0) < \overline{\lim}(f, x_0^-)$, for some $x_0 \in \mathbb{R}$. Without loss of generality, we can assume that the first inequality holds. Define $g = -\chi_\mathbb{R} \in \acute{\mathcal{S}}_{es} lsc$. Notice that $fg = -f$ on $\mathbb{R}$ and

$$\underline{\lim}(fg, x_0^+) = \underline{\lim}(-f, x_0^+) = -\overline{\lim}(f, x_0^+) < -f(x_0) = (fg)(x_0),$$

whence $fg \notin lsc$. Consequently, $f \notin \mathcal{M}_m(\acute{\mathcal{S}}_{es} lsc, \acute{\mathcal{S}}_s lsc)$, and we may assume that $f \in \mathcal{C}$. We consider two cases.

<u>*Case 1*</u>. $f(x_0) < 0$, for some $x_0 \in \mathbb{R}$.

Then $f < 0$ on $(x_0 - \delta, x_0 + \delta)$, for some $\delta > 0$. Define

$$g(x) = \begin{cases} \big(\sin(x - x_0)\big)^{-1}, & \text{if } x \neq x_0, \\ -1, & \text{if } x = x_0. \end{cases}$$

We can easily see that $g \in \acute{\mathcal{S}}_{es} lsc$. Moreover, $fg \in usc \setminus \mathcal{C}$ at x_0, whence $fg \notin lsc$. So, in this case $f \notin \mathcal{M}_m(\acute{\mathcal{S}}_{es} lsc, \acute{\mathcal{S}}_s lsc)$.

<u>*Case 2*</u>. $f(x) \geq 0$, for each $x \in \mathbb{R}$.

Then there is a closed interval $[a,b]$ such that f is positive and nonconstant on $[a,b]$. By Theorem 3.1, there is a function $\bar{g}\colon [a,b] \to \mathbb{R}$ such that $\bar{g} \in \acute{\mathcal{S}}_{es}lsc$ and $\ln \circ f + \bar{g} \notin \acute{\mathcal{S}}_{s}lsc$ on $[a,b]$. Define

$$g(x) = \begin{cases} (\exp \circ \bar{g})(x), & \text{if } x \in [a,b], \\ (\exp \circ \bar{g})(a), & \text{if } x \in (-\infty, a), \\ (\exp \circ \bar{g})(b), & \text{if } x \in (b, \infty). \end{cases}$$

By Lemmas 2.5 and 2.6, $\exp \circ \bar{g} \in \acute{\mathcal{S}}_{es}lsc$ on $[a,b]$, whence clearly $g \in \acute{\mathcal{S}}_{es}lsc$. But on the interval $[a,b]$, we have

$$\ln \circ (fg) = \ln \circ \big(f \cdot (\exp \circ \bar{g})\big) = \ln \circ f + \bar{g} \notin \acute{\mathcal{S}}_{s}lsc.$$

If $fg \in \acute{\mathcal{S}}_{s}lsc$ on $[a,b]$ then by Lemmas 2.4 and 2.6, $\ln \circ (fg) \in \acute{\mathcal{S}}_{s}lsc$ on $[a,b]$, a contradiction. So, $fg \notin \acute{\mathcal{S}}_{s}lsc$, which proves that $f \notin \mathcal{M}_m(\acute{\mathcal{S}}_{es}lsc, \acute{\mathcal{S}}_{s}lsc)$. This completes the proof of condition (1).

Now, we will prove that $\mathcal{M}_m(\acute{\mathcal{S}}_{es}usc, \acute{\mathcal{S}}_{s}usc) \subset \mathcal{C}onst_+ \cup \{\chi_\varnothing\}$. Let the function $f \notin \mathcal{C}onst_+ \cup \{\chi_\varnothing\}$. By condition (1), $f \notin \mathcal{M}_m(\acute{\mathcal{S}}_{es}lsc, \acute{\mathcal{S}}_{s}lsc)$. Hence, there is a function $\bar{g} \in \acute{\mathcal{S}}_{es}lsc$ such that $f\bar{g} \notin \acute{\mathcal{S}}_{s}lsc$. Put $g = -\bar{g}$. Then $g \in \acute{\mathcal{S}}_{es}usc$ and $-(fg) = f\bar{g} \notin \acute{\mathcal{S}}_{s}lsc$. So, $fg \notin \acute{\mathcal{S}}_{s}usc$, whence $f \notin \mathcal{M}_m(\acute{\mathcal{S}}_{es}usc, \acute{\mathcal{S}}_{s}usc)$. Since

$$\mathcal{M}_m(\acute{\mathcal{S}}_{es}lsc, \acute{\mathcal{S}}_{s}lsc) \subset \mathcal{C}onst_+ \cup \{\chi_\varnothing\} \subset \mathcal{M}_m(\acute{\mathcal{S}}_{s}lsc) \cap \mathcal{M}_m(\acute{\mathcal{S}}_{es}lsc)$$

and

$$\mathcal{M}_m(\acute{\mathcal{S}}_{es}usc, \acute{\mathcal{S}}_{s}usc) \subset \mathcal{C}onst_+ \cup \{\chi_\varnothing\} \subset \mathcal{M}_m(\acute{\mathcal{S}}_{s}usc) \cap \mathcal{M}_m(\acute{\mathcal{S}}_{es}usc),$$

using Remark 1.1, we obtain that

$$\mathcal{M}_m(\acute{\mathcal{S}}_{s}lsc) = \mathcal{M}_m(\acute{\mathcal{S}}_{es}lsc) = \mathcal{M}_m(\acute{\mathcal{S}}_{s}usc) = \mathcal{M}_m(\acute{\mathcal{S}}_{es}usc) = \mathcal{C}onst_+ \cup \{\chi_\varnothing\}. \blacksquare$$

Theorem 3.3. $\mathcal{M}_{\max}(\acute{\mathcal{S}}_{s}lsc) = \mathcal{M}_{\max}(\acute{\mathcal{S}}_{es}lsc) = \mathcal{C}onst.$

Proof. The proof of the inclusion $\mathcal{M}_{\max}(\acute{\mathcal{S}}_{es}lsc, \acute{\mathcal{S}}_{s}lsc) \subset \mathcal{C}onst$ is similar to the proof of [5, Theorem 3.5]. Since $\mathcal{C}onst \subset \mathcal{M}_{\max}(\acute{\mathcal{S}}_{s}lsc) \cap \mathcal{M}_{\max}(\acute{\mathcal{S}}_{es}lsc)$, using Remark 1.1, we obtain that $\mathcal{M}_{\max}(\acute{\mathcal{S}}_{s}lsc) = \mathcal{M}_{\max}(\acute{\mathcal{S}}_{es}lsc) = \mathcal{C}onst$. ■

Theorem 3.4. *The function $f \in \mathcal{M}_{\max}(\acute{\mathcal{S}}_{es}usc)$ if and only if $f \in \acute{\mathcal{S}}_{es}usc$ and two following conditions hold*:

(2) *for each $x \notin \mathcal{C}^+(f)$, there is a $\delta > 0$ such that $f(t) \leq f(x)$, for each $t \in (x-\delta, x)$,*

(3) *for each $x \notin \mathcal{C}^-(f)$, there is a $\delta > 0$ such that $f(t) \leq f(x)$, for each $t \in (x, x+\delta)$.*

Proof. First, assume that $f \in \acute{\mathcal{S}}_{es}usc$ and conditions (2) and (3) are fulfilled. We will show that $f \in \mathcal{M}_{\max}(\acute{\mathcal{S}}_{es}usc)$. Fix a function $g \in \acute{\mathcal{S}}_{es}usc$ and let $h = \max\{f, g\}$. Since the maximum of two upper semicontinuous functions is upper semicontinuous (see e.g. [2, p. 83]), $h \in usc$. So, we must show that $h \in \acute{\mathcal{S}}_{es}$.

Let $\alpha < \beta$ and $y \in \mathrm{I}[h(\alpha), h(\beta)]$. Assume that $h(\alpha) \le h(\beta)$. (The case $h(\alpha) \ge h(\beta)$ is analogous.) Since $\mathcal{M}_{\max}(\mathcal{D}) = \mathcal{D}usc$ [1], we have $h \in \mathcal{D}$, whence $h(x_0) = y$, for some $x_0 \in [\alpha, \beta]$.

If $f(x_0) > g(x_0)$ then since $f \in \acute{\mathcal{S}}_{es}$ and $g \in \acute{\mathcal{S}}_{es}usc \subset \acute{\mathcal{S}}_{s}usc$, by Lemma 2.2 and Remark 2.7, there is an $x_1 \in [\alpha, \beta] \cap \mathcal{C}(f)$ such that $g(x_1) < f(x_1) = f(x_0)$. Using the fact that $h = \max\{f, g\}$, we clearly obtain that $x_1 \in \mathcal{C}(h)$ and $h(x_1) = f(x_1) = f(x_0) = h(x_0) = y$.

If $g(x_0) > f(x_0)$, we proceed analogously. So, let $f(x_0) = g(x_0) = h(x_0) = y$. If $x_0 \in \mathcal{C}(f)$ then since $g \in \acute{\mathcal{S}}_{es}usc$, we have $x_0 \in \mathcal{C}(h)$. So, we can assume that $x_0 \notin \mathcal{C}(f)$, whence $x_0 \notin \mathcal{C}^+(f)$ or $x_0 \notin \mathcal{C}^-(f)$. Suppose that e.g. $x_0 \notin \mathcal{C}^+(f)$. (The other case is analogous.) First let $x_0 = \alpha$. We consider two cases.

<u>*Case 1*</u>. $x_0 \notin \mathcal{C}^-(f)$.

Then, by the assumption (3), there is a $\delta > 0$ such that $f(t) \le f(x_0)$, for each $t \in (x_0, x_0 + \delta)$. But since $g \in \acute{\mathcal{S}}_{es}$, there is an $x_1 \in [x_0, x_0 + \delta) \cap \mathcal{C}(g) \cap [\alpha, \beta]$ with $g(x_1) = y$. So, since $f \in \acute{\mathcal{S}}_{es}usc$, $f(x_1) \le g(x_1)$, and $h = \max\{f, g\}$, we have $x_1 \in [\alpha, \beta] \cap \mathcal{C}(h)$ and $h(x_1) = y$.

<u>*Case 2*</u>. $x_0 \in \mathcal{C}^-(f)$.

If there is a $\tau > 0$ such that $h(t) \ge h(x_0)$, for each $t \in (x_0, x_0 + \tau)$ then $x_0 \in \mathcal{C}(h)$. In the other case, choose a $\tau > 0$ such that $x_0 + \tau < \beta$. There is a $t_\tau \in (x_0, x_0 + \tau)$ with $h(t_\tau) < h(x_0)$. Define

$$t_0 = \sup\{t \in [x_0, t_\tau) : \; h(t) = h(x_0)\}.$$

The fact $h \in \mathcal{D}$ implies that

$$h(x) < h(x_0), \quad \text{for each } x \in (t_0, t_\tau). \tag{4}$$

Moreover, observe that since $h \in \mathcal{D}$ and condition (4) holds, we have $h(t_0) \le h(x_0)$. But $h \in usc$, whence $h(t_0) = h(x_0)$. So, since $f, g \in \acute{\mathcal{S}}_{es}$, $h = \max\{f, g\}$, and condition (4) holds, $t_0 \in \mathcal{C}(h)$. Consequently, $t_0 \in [\alpha, \beta] \cap \mathcal{C}(h)$ and $h(t_0) = y$.

Finally, let $x_0 \in (\alpha, \beta]$. Then, by the assumption (2), there is a $\delta > 0$ such that $f(t) \le f(x_0)$, for each $t \in (x_0 - \delta, x_0)$. But since $g \in \acute{\mathcal{S}}_{es}$, there is an $x_1 \in (x_0 - \delta, x_0] \cap \mathcal{C}(g) \cap [\alpha, \beta]$ with $g(x_1) = y$. So, since $f \in \acute{\mathcal{S}}_{es}usc$, $f(x_1) \le g(x_1)$, and $h = \max\{f, g\}$, we have $x_1 \in [\alpha, \beta] \cap \mathcal{C}(h)$ and $h(x_1) = y$.

So, $h \in \acute{\mathcal{S}}_{es}usc$, which proved that $f \in \mathcal{M}_{\max}(\acute{\mathcal{S}}_{es}usc)$.

Now, we will show that $\mathcal{M}_{\max}(\acute{\mathcal{S}}_{es}usc) \subset \acute{\mathcal{S}}_{es}usc$. Let $f \notin \acute{\mathcal{S}}_{es}usc$. First assume that $f \notin \acute{\mathcal{S}}_{es}$. Then there are $\alpha < \beta$ and $y \in \mathrm{I}[f(\alpha), f(\beta)]$ such that $f(x) \neq y$, for each $x \in [\alpha, \beta] \cap \mathcal{C}(f)$. Put $g = \min\{f(\alpha) - 1, f(\beta) - 1\}$ and $h = \max\{f, g\}$. Then clearly $g \in \mathcal{C}onst \subset \acute{\mathcal{S}}_{es}usc$. Since $y \in \mathrm{I}[h(\alpha), h(\beta)]$ and $h(x) \neq y$, for each $x \in [\alpha, \beta] \cap \mathcal{C}(h)$, we have $h \notin \acute{\mathcal{S}}_{es}$. So, $h \notin \acute{\mathcal{S}}_{es}usc$, whence $f \notin \mathcal{M}_{\max}(\acute{\mathcal{S}}_{es}usc)$.

Now assume that $f \notin usc$. Then e.g., $f(x_0) < \overline{\lim}(f, x_0^+)$, for some $x_0 \in \mathbb{R}$. (The other case is analogous.) Put $g = f(x_0)$ and $h = \max\{f, g\}$. Then clearly $g \in \mathcal{C}onst \subset \acute{\mathcal{S}}_{es}usc$, and since

$$h(x_0) = g(x_0) = f(x_0) < \overline{\lim}(f, x_0^+) = \overline{\lim}(h, x_0^+),$$

$h \notin usc$. So, $h \notin \acute{\mathcal{S}}_{es}usc$, whence $f \notin \mathcal{M}_{\max}(\acute{\mathcal{S}}_{es}usc)$.

To complete the proof, we assume that $f \in \acute{\mathcal{S}}_{es}usc$ and condition (3) is not fulfilled. (Similarly, we can proceed if $f \in \acute{\mathcal{S}}_{es}usc$ and condition (2) does not hold.) Then there is an $x_0 \notin \mathcal{C}^-(f)$ and we can choose a sequence (x_n) such that $x_n \to x_0^+$ and $f(x_n) > f(x_0)$, for each $n \in \mathbb{N}$. Since $f \in \acute{\mathcal{S}}_{es}$, we may assume that $(x_n) \subset \mathcal{C}(f)$. Hence for each $n \in \mathbb{N}$, there is a $\delta_n > 0$ such that $f(x) > f(x_0)$, for each $x \in [x_n - \delta_n, x_n + \delta_n]$. Without loss of generality we can assume that $x_{n+1} + \delta_{n+1} < x_n - \delta_n$, for each $n \in \mathbb{N}$. Define

$$g(x) = \begin{cases} f(x), & \text{if } x \in (-\infty, x_0], \\ f(x_0), & \text{if } x \in \bigcup_{n=1}^{\infty}[x_n - \delta_n, x_n + \delta_n] \cup (x_1 + \delta_1, \infty), \\ f(x_0) + n^{-1}, & \text{if } x = c_n,\ n \in \mathbb{N}, \\ \text{linear}, & \text{in each interval } [x_{n+1} + \delta_{n+1}, c_n] \text{ and } [c_n, x_n - \delta_n], \\ & n \in \mathbb{N}, \end{cases}$$

where

$$c_n = \frac{x_{n+1} + \delta_{n+1} + x_n - \delta_n}{2}.$$

Then clearly $g \in usc$. Moreover, since $f \in \acute{\mathcal{S}}_{es}$ and $g \restriction (x_0, \infty) \in \mathcal{C}$, by Lemma 2.2, $g \in \acute{\mathcal{S}}_{es}$. Now we will show that $h = \max\{f, g\} \notin \acute{\mathcal{S}}_{es}$.

Put $\alpha = x_0$ and $\beta = c_1$. Notice that $x_0 \notin \mathcal{C}(h)$. Now fix an $x \in (\alpha, \beta]$. Then $h(x) > f(x_0)$. Indeed, if $x \in [x_n - \delta_n, x_n + \delta_n]$ for some $n \in \mathbb{N}$ then $h(x) \geq f(x) > f(x_0)$, and if $x \in (x_{n+1} + \delta_{n+1}, x_n - \delta_n)$, for some $n \in \mathbb{N}$ then

$$h(x) \geq g(x) > g(x_0) = f(x_0).$$

Hence in particular $f(x_0) \in [h(\alpha), h(\beta)]$ and $h(x) \neq f(x_0)$, for each $x \in [\alpha, \beta] \cap \mathcal{C}(h)$. Therefore $h \notin \acute{\mathcal{S}}_{es}$, whence $f \notin \mathcal{M}_{\max}(\acute{\mathcal{S}}_{es}usc)$. This competes the proof. ■

An immediate consequence of Theorem 3.4 is the following corollary.

Corollary 3.5. $\mathcal{C} \subset \mathcal{M}_{\max}(\acute{\mathcal{S}}_{es}usc)$.

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