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An "Elementary" Calculation of the Krull Dimension of a Polynomial Ring over a Field

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1. Introduction It is well known that the Krull dimension of a domain R, which is finitely generated over a field K, is equal to the transcendence degree over K of its field of fractions; that is

$$\dim(R) = \text{tr.deg}_K(R_0)$$

(see, for example, [1]). In the proof, the central role plays Noether normalization lemma (see [3]). Namely, if $\text{tr.deg}_K(R_0) = r$, then R is an integral extension of a subring $S = K[x_1, \ldots, x_r]$, where $x_1, \ldots, x_r$ are algebraically independent over K. In particular, S is isomorphic to $K[T_1, \ldots, T_r]$, the polynomial ring in r variables over K. The rest of the proof is a consequence of the following two facts:

(1) $\dim(K[T_1, \ldots, T_r]) = r$,

(2) if R is an integral extension of S then $\dim(R) = \dim(S)$.

The proof of (2) is not direct but, in fact, quite elementary ("going up" theorem, see [1]). In contrast, the proof of (1) involves some non-elementary methods, like principal ideal theorem, systems of parameters, or Hilbert-Samuel polynomials (see [2], [1]). Our proof is based only on well known facts including the above normalization lemma, and was first published in the manuscript [4].

2. The proof of (1) We prove by induction on n that $\dim(S) = n$, where $S = K[T_1, \ldots, T_n]$. The chain of prime ideals

$$O \subset (T_1) \subset (T_1, T_2) \subset \ldots \subset (T_1, \ldots, T_n) \tag{1}$$

shows that $\dim(S) \geq n$. Suppose that $\dim(S) > n$. Then there exists a chain of prime ideals

$$P_{-1} \subset P_0 \subset \ldots \subset P_n$$

of the ring S. Since P_0 is non-zero, it contains a non-zero and non - invertible polynomial g. One of its indecomposable factors, say f, does not belong to the prime ideal P_0. Since S is a UFD, f generates a prime ideal, and hence $R = S/f$ is a domain. It follows from the choice of f that $\dim(R) \geq n$. On the other hand, $S = K[t_1, \ldots, t_n]$, where t_i denotes the coset of T_i for $i = 1, \ldots, n$, and then the relation $f(t_1, \ldots, t_n) = 0$ gives us $\mathrm{tr.deg}_k(R_0) = r < n$. Consequently, R is an integral extension of $K[x_1, \ldots, x_r]$ for some algebraically independent elements $x_1, \ldots, x_r$. By induction, $\dim(K[T_1, \ldots, T_r]) = r$, and hence (2) gives us $\dim(R) = r < n$, a contradiction.

3. Corollary It follows from (0.1) that $\mathrm{ht}(T_1, \ldots, T_i) \geq i$. We give the proof of the equality

$$\mathrm{ht}(T_1, \ldots, T_i) = i \quad \text{for} \quad i = 1, \ldots, n$$

omitting the principal ideal theorem.

Suppose that $\mathrm{ht}(T_1, \ldots, T_i) > i$ for some i. Then there exists a chain of prime ideals

$$P_0 \subset \ldots \subset P_i \subset (T_1, \ldots, T_i) \subset \ldots \subset (T_1, \ldots, T_n),$$

what gives us $\dim(K[T_1, \ldots, T_n]) > n$, a contradiction.

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