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An Absorption Property for Almost Continuous and Quasi-continuous Functions

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In many papers (see e.g. [4], [5], [6] [10], [11]), the authors considered a lot of problems connected with compositions of *almost continuous* functions (in the sense of Stallings). In particular, it was proved ([5]), that there exists an almost continuous function $f : I \longrightarrow I$ ($I = [0,1]$) such that $f \circ f$ is not an almost continuous function. With reference to the considerations included in paper [2] and [12] we can ask the question: What additional assumptions must a surjection $f : I \longrightarrow I$ satisfy in order that, for each function $g : I \longrightarrow I$, the implication:

if $g \circ f$ is almost continuous then g is almost continuous

be true. (In the paper [6] the authors prove that there exist a continuous surjection: $f : I \longrightarrow I$ and an almost continuous function $g : I \longrightarrow I$ such that $g \circ f$ is almost continuous).

It is also of some interest to ask similar question for the case of quasi-continuous functions.

The problems seem to be very interesting, among others, on account Theorem 1 presented bellow wich states that, in the space of functions with the metric of uniform convergence in each neighbourhood of any homeomorphism[1], we can find a continuum of quasi-continuous and almost continuous functions of the first class of Baire which do not possess the right absorption property (with respect to the family of almost continuous functions as well as quasi-continuous functions) and we can also find a continuum of functions which have this property.

Throughout the paper we use the classical symbols and notations. In particular I denotes the unit interval with the natural topology. To simplify the notations, instead of $[a,b] \cap I$ ($(a,b) \cap I$ etc.) we shall often write in short: $[a,b]$ ((a,b) etc.).

The graph of the function f will be denoted by $\mathcal{G}(f)$.

A function $f: X \longrightarrow Y$ is *almost continuous* (in the sense of Stallings) if for each open set $U \subset X \times Y$ containing $\mathcal{G}(f)$, U contains a graph of some continuous function. The class of all almost continuous functions $f: I \longrightarrow I$ will be denoted by $\mathcal{A}$.

If $F \subset X \times Y$ is a closed set such that $F \cap \mathcal{G}(f) = \emptyset$ and $F \cap \mathcal{G}(g) \neq \emptyset$ for every continuous function, then F is called a *blocking set* for f.

We say that a function $f: X \longrightarrow Y$ is *quasi-continuous* at a point $x_0 \in X$ if, for any neighbourhoods U of the point $f(x_0)$ and V of x_0, $\operatorname{Int}(f^{-1}(U) \cap V) \neq \emptyset$ (by the symbol $\operatorname{Int}(A)$ we denote the interior of the set A). If a function f is quasi-continuous at each point of its domain, then we say that f is quasi-continuous. The set of all quasi-continuous functions $f: I \longrightarrow I$ will be denoted by $\mathcal{Q}_c$.

By the symbol $\mathcal{B}_1\mathcal{A}$ we shall denote the subset of $\mathcal{A}$ consiting of all functions in Baire class one.

Applying the idea contained in paper [2], we can assume the following definition. Let $\mathcal{F}$ be a fixed family of functions mapping I into I. We say that a surjection $f: I \xrightarrow{onto} I$ has a *right absorption property* relative to $\mathcal{F}$ (abbreviated $f \in RAP(\mathcal{F})$)[2], provided that if $g: I \longrightarrow I$ is a function such that $g \circ f \in \mathcal{F}$, then $g \in \mathcal{F}$.

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[1] As Theorem 2 will show, in our considerations homeomorphisms will play an essential role while we are looking for properties distinguishing functions which possess the right absorption property relative to the families $\mathcal{A}$ and $\mathcal{Q}_c$

[2] $RAP(\mathcal{F})$ will thus denote the set of all surjections possessing the right absorption property relative to the $\mathcal{F}$.

If $\mathcal{F}$ is some family of functions, then by $(\mathcal{F})_s$ we shall denote the subset (subspace), composed of those functions of the family $\mathcal{F}$, which are surjections.

The question raised at the beginning of the paper can be formulated in the following way: Under what additional assumptions does a function f belong to $RAP(\mathcal{B}_1\mathcal{A})$ (or $RAP(\mathcal{Q}_c)$). Knowing the theorem ([3]: If $h : I \xrightarrow{onto} I$ is a homeomorphism and $f \in \mathcal{A}$, then $f \circ h \in \mathcal{A}$) and the results contained in paper [12], it is easy to see that in this case, it can be useful to distinguish the class $\widehat{H}$ of surjections from I onto itself.

Definition 1 *A surjection $f : I \xrightarrow{onto} I$ belongs to the family $\widehat{H}$ if, for an arbitrary element $\alpha \in I$, there exist real numbers $a, b \in I$ such that $f_{|[a,b]}$ is a homeomorphism and $\alpha \in \operatorname{Int}(f([a,b]))$.*[3]

The basic considerations will be preceded by the lemmas[4], which will be used in the further part of the paper.

Lemma 1 [13] *If A is a closed subset of X and $f : X \longrightarrow Y$ is an almost continuous function, then the function $f_{|A}$ is almost continuous.*

Lemma 2 [11] *Let an interval J be a union of countably many closed intervals I_n such that $\operatorname{Int}(I_n) \cap \operatorname{Int}(I_m) = \emptyset$ for $m \neq n$ and $I_n \cap I_{n+1} \neq \emptyset$ for every positive integer n, and let Y_0 be a convex subset of a normed space Y. For any function $f : J \longrightarrow Y_0$, f is almost continuous if and only if $f_{|I_n}$ is almost continuous for every n.*

Lemma 3 *A function $f : I \longrightarrow I$ is not almost continuous if and only if there exists a point x_0 such that $f_{|[x_0-\sigma, x_0+\sigma]}$ is not almost continuous for each $\sigma > 0$.*

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[3]Of course, $\widehat{H} \setminus (\mathcal{A} \cup \mathcal{Q}_c) \neq \emptyset$.

[4]The facts concerning quasi-continuity, used in the proof of Theorem 1 are so simple that they require no additional explanations in the form of lemmas or detailed proofs.

Proof. Sufficiency follows from Lemma 1.

Necessity. Let I_1 denote an interval chosen from $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$ in the way that $f_{|I_1}$ is not almost continuous (the existence of such interval follows from Lemma 2). Let us divide the interval I_1 into two closed intervals of lenght $\frac{1}{4}$ and denote by I_2 the one for which $f_{|I_2}$ is not almost continuous. Proceeding further in this way, we shall obtain a decreasing sequence of intervals $\{I_n\}_{n=1}^{\infty}$, such that the diameter of I_n tends to zero. Let $\{x_0\} = \bigcap_{n=1}^{\infty} I_n$. Then x_0 is the required point.

Lemma 4 [1] *$f \in \mathcal{B}_1\mathcal{A}$ if and only if f is a Darboux function in Baire class one.*

Lemma 5 ([11], [13]) *For each $f \in \mathcal{A}$ and a continuous function g, the composition $g \circ f$ is almost continuous.*

It was shown in paper [2] that $(\mathcal{C})_s \subset RAP(\mathcal{C})$, where $\mathcal{C}$ is the class of all continuous functions. Simple examples show that, for the families $\mathcal{B}_1\mathcal{A}$ and $\mathcal{Q}_c$ analogous inclusions do not hold. Moreôver, from the theorem below we can conlude that every homeomorphism can be uniformly approximated by functions from the class $\mathcal{B}_1\mathcal{A} \cap \mathcal{Q}_c$, which do not possess the right absorption property.

Theorem 1 (a) *In the space of all functions $f : I \longrightarrow I$ with the metric of uniform convergence, the set*

$$(\mathcal{B}_1\mathcal{A} \cap \mathcal{Q}_c)_s \setminus (RAP(\mathcal{A}) \cup RAP(\mathcal{Q}_c))$$

is c-dense in the set H of all homeomorphisms $h : I \xrightarrow{onto} I$.[5]

(b) *In the space of all functions $f : I \longrightarrow I$ with the metric of uniform convergence, a set $(\mathcal{B}_1\mathcal{A} \cap \mathcal{Q}_c)_s \cap RAP(\mathcal{A}) \cap RAP(\mathcal{Q}_c)$ is a c-dense set for the set H of all homeomorphisms $h : I \xrightarrow{onto} I$.*

Proof. Denote

$$\mathcal{A}^* = (\mathcal{B}_1\mathcal{A} \cap \mathcal{Q}_c)_s \setminus (RAP(\mathcal{A}) \cup RAP(\mathcal{Q}_c))$$

[5] i.e. for each $h \in H$ and a neighbourhood V of h, the cardinality of $V \cap [(\mathcal{B}_1\mathcal{A} \cap \mathcal{Q}_c) \setminus (RAP(\mathcal{A}) \cup RAP(\mathcal{Q}_c))]$ is not less than continuum.

and

$$\mathcal{Q}_c^* = (\mathcal{B}_1\mathcal{A} \cap \mathcal{Q}_c)_s \cap RAP(\mathcal{A}) \cap RAP(\mathcal{Q}_c).$$

Let $h \in H$ and $\varepsilon > 0$ (to simplify the further notation, we assume that $\varepsilon < 1$ and h is an increasing function).

Let $[p,q] \subset (0,1)$ be a segment such that $h([p,q]) \subset (\frac{1}{2} - \frac{\varepsilon}{2}, \frac{1}{2} + \frac{\varepsilon}{2})$ and $h(p) < \frac{1}{2} < h(q)$. We shall show that

(1) for each $\alpha \in (h(p), h(q))$ there exist functions $f_\alpha \in \mathcal{A}^*$ such that $sup_{x\in I} \mid h(x) - f_\alpha(x) \mid < \varepsilon$ and $f_{\alpha_1} \neq f_{\alpha_2}$ for $\alpha_1 \neq \alpha_2$.

(1′) for each $\alpha \in (h(p), h(q))$ there exist functions $k_\alpha \in \mathcal{A}^*$ such that $sup_{x\in I} \mid h(x) - k_\alpha(x) \mid < \varepsilon$ and $k_{\alpha_1} \neq k_{\alpha_2}$ for $\alpha_1 \neq \alpha_2$.

First, we construct the functions f_α for $\alpha \in (\frac{1}{2} - \frac{\varepsilon}{2}, \frac{1}{2} + \frac{\varepsilon}{2})$ (proof of part (a)).

Denote by z_0 the midpoint of $[p,q]$ and let $\alpha_o \in (h(p), h(q))$. Finally, let $\{s_n\}_{n=1}^\infty \subset (p, z_0)$ and $\{z_n\}_{n=1}^\infty \subset (z_0, q)$ denote arbitrary sequences such that $s_n \nearrow z_0 \swarrow z_n$.

Now, we define a function $f_{\alpha_0} : I \xrightarrow{onto} I$ in the following way:

$$f_{\alpha_0}(x) = \begin{cases} \alpha_o & \text{if } x = z_0, \\ h(x) & \text{if } x \in [0,p] \cup [q,1], \\ \frac{1}{2} + \frac{\varepsilon}{2} & \text{if } x = z_{2n-1} \text{ for } n \in \mathsf{N}, \\ \alpha_o + \min(\frac{\varepsilon+1}{2} - \alpha_o, \frac{1}{2n}) & \text{if } x = z_{2n} \text{ and } n \in \mathsf{N}, \\ \frac{1}{2} - \frac{\varepsilon}{2} & \text{if } x = s_{2n-1} \text{ and } n \in \mathsf{N}, \\ \alpha_o - \min(\frac{1-\varepsilon}{2} + \alpha_o, \frac{1}{2n}) & \text{if } x = s_{2n} \text{ and } n \in \mathsf{N}, \\ linear & \text{in each of the segments } [s_n, s_{n+1}], [z_{n+1}, z_n],\ n \in \mathsf{N} \text{ and } [p, s_1], [z_1, q]. \end{cases}$$

It is easy to see (the property connected with almost continuity follows from the above lemmas) that $f_{\alpha_0} \in (\mathcal{B}_1\mathcal{A})_s \cap (\mathcal{Q})_s$ and

$$\sup_{x\in I} \mid h(x) - f_{\alpha_0}(x) \mid < \varepsilon.$$

To finish the proof of (a), it is sufficient to show that

(2) $f_{\alpha_0} \notin RAP(\mathcal{A}) \cup RAP(\mathcal{Q}_c)$.

Let $g : I \longrightarrow I$ be defined by the formula:

$$g(x) = \begin{cases} 1 & \text{if } x \in \{\alpha_o\} \cup [0, \frac{1}{2} - \frac{\varepsilon}{2}) \cup (\frac{1}{2} + \frac{\varepsilon}{2}, 1]; \\ \frac{2\cdot(x-\alpha_o)}{\varepsilon+1-2\cdot\alpha_o} & \text{if } x \in (\alpha_o, \frac{1}{2} + \frac{\varepsilon}{2}]; \\ \frac{2\cdot(\alpha_o - x)}{2\cdot\alpha_o - 1 + \varepsilon} & \text{if } x \in [\frac{1}{2} - \frac{\varepsilon}{2}, \alpha_o). \end{cases}$$

Of course, α_o is not a Darboux point[6] of g, and so, $g \notin \mathcal{A}$ (see [13], [11]) and, in an obvious way, we may deduce that $g \notin \mathcal{Q}_c$.

Now, we consider the function $\xi = g \circ f_{\alpha_0}$. According to lemmas 1 and 5, $\xi_{|[0,p]}$ and $\xi_{|[q,1]}$ are almost continuous functions (and, moreover, it is not hard to verify that $\xi_{|[0,p]}$ and $\xi_{|[q,1]}$ are quasi-continuous functions). Remark that $\xi_{|[p,q]}$ posesses exactly one point of discontinuity, namely z_0. According to lemmas 4 and 2, to prove (2), it suffices to show that

(3) z_0 is a Darboux point and a quasi-continuity point of $\xi^* = \xi_{|[p,q]}$.

So, let β be an arbitrary right-sided cluster number of ξ^* at z_0 (in the case of a left-sided cluster number, the proof is similar) such that $\beta < 1 = \xi^*(z_0)$ and let $\gamma \in (\beta, 1)$ and $\sigma > 0$. Then there exists a positive integer n_o such that $z_{2n_o-1} \in [z_0, z_0 + \sigma)$ and $f_{\alpha_o}(z_{2n_o}) < t^*$, where $\{t^*\} = g^{-1}(\gamma) \cap (\alpha_o, \frac{1}{2} + \frac{\varepsilon}{2})$. Then $\xi^*_{|[z_{2n_o}, z_{2n_o-1}]}$ is a continuous function such that $\xi^*(z_{2n_o}) < \gamma$ and $\xi^*(z_{2n_o-1}) = 1$ and, consequently, $\xi^*(t_o) = \gamma$ for some $t_o \in (z_{2n_o}, z_{2n_o-1}) \subset [z_0, z_0 + \sigma)$, and t_o is a continuity point of ξ, what ends the proof of (3).

Now, we shall prove part (b) of the theorem. Let $x_0, y_0, z_0 \in (p, q)$ be points such that $x_0 < y_0 < z_0$ and let $\alpha_0 \in (h(p), h(q))$. Define the

[6]In this proof we shall make use of the form of the definition of *Darboux point* proposed by J.S. Lipiński [7]: Let $f : I \longrightarrow I$. A point $t_o \in I$ is said to be a right-sided (left-sided) Darboux point of f provided that for each $\sigma > 0$ and for each number β which is strictly between $f(t_o)$ and some right-sided (left-sided) cluster number of f at t_o, there exists $z \in [t_o, t_o + \sigma)$ ($z \in (t_o - \sigma, t_o]$) such that $f(z) = \beta$. The point t_o is a *Darboux point* of f, provided t_o is a right-sided and left-sided Darboux point of f.

function k_{α_0} in the following way:

$$k_{\alpha_0}(x) = \begin{cases} h(x) & \text{if } x \in [0,p] \cup [q,1], \\ \alpha_0 & \text{if } x = x_0, \\ \frac{1}{2} + \frac{\varepsilon}{2} & \text{if } x = y_0, \\ \frac{1}{2} - \frac{\varepsilon}{2} & \text{if } x = z_0, \\ \textit{linear} & \text{in the contagious intervals.} \end{cases}$$

It is obvious that $k_{\alpha_0} \in \mathcal{B}_1\mathcal{A} \cap \mathcal{Q}_c \cap RAP(\mathcal{A}) \cap RAP(\mathcal{Q}_c)$.

Let us now proceed to answering the basic question raised at the beginning of the paper.

Theorem 2 $\widehat{H} \subset RAP(\mathcal{A}) \cap RAP(\mathcal{Q}_c)$.

Proof. Let $f \in \widehat{H}$.

First, we suppose that $f \notin RAP(\mathcal{A})$. This means that there exists a function $g : I \longrightarrow I$ such that

(4) $g \circ f \in \mathcal{A}$ and $g \notin \mathcal{A}$.

From Lemma 3 we conclude that there exists $x_0 \in I$ such that

(5) $g_{|[x_0-\sigma, x_0+\sigma]}$ is not an almost continuous function for $\sigma > 0$.

So, let $[a,b] \subset I$ be an interval such that $h = f_{|[a,b]}$ is a homeomorphism and $x_0 \in \operatorname{Int}(h([a,b]))$. Denote by y_0 a point from the interval $[a,b]$ such that $h(y_0) = x_0$ and let $\delta > 0$ be a real number for which $[y_0 - \delta, y_0 + \delta] \subset [a,b]$ and $h([y_0 - \delta, y_0 + \delta]) \subset \operatorname{Int}(h([a,b]))$. Let $[\alpha, \beta] = h([y_0 - \delta, y_0 + \delta])$. Then $x_0 \in \operatorname{Int}([\alpha,\beta])$[7]. Obviously (according to (5)), the function $g^* = g_{|[\alpha,\beta]}$ is not almost continuous and, at the same time, there exists a blocking set $F \subset [\alpha,\beta] \times I$ for the function g^*.

Now, we consider the set

$$F^* = \{(p,q) \in [y_0 - \delta, y_0 + \delta] \times I : \ (h(p), q) \in F\}.$$

We shall show that

[7]Our considerations also include the case when, e.g. $x_0 = 0$, hence $\operatorname{Int}([\alpha,\beta])$ cannot be replaced by (α, β)

(6) F^* is a blocking set for $h^* = g^* \circ h_{|[y_0-\delta, y_0+\delta]}$.

First, we observe that

(7) $\mathcal{G}(h^*) \cap F^* = \emptyset$.

Indeed, suppose to the contrary that there exist $p_o \in [y_0 - \delta, y_0 + \delta]$ and $q_o \in I$ such that $(p_o, q_o) \in \mathcal{G}(h^*) \cap F^*$. Since $(p_o, q_o) \in F^*$, then $(h(p_o), q_o) \in F$ and, by $(p_o, q_o) \in \mathcal{G}(h^*)$, we deduce that

$$(h(p_o), q_o) \in \mathcal{G}(g^*),$$

what contradicts to the fact that F is a blocking set for g^*. Equality (7) is proved.

It is evident that

(8) F^* is a closed set.

Now, we shall prove that:

(9) $\mathcal{G}(\xi) \cap F^* \neq \emptyset$ for any continuous function $\xi : [y_0 - \delta, y_0 + \delta] \longrightarrow I$.

Indeed. Put $\xi' = \xi \circ h^{-1}_{|[\alpha,\beta]}$. Then $\mathcal{G}(\xi') \cap F \neq \emptyset$. Let $(\varphi, \psi) \in \mathcal{G}(\xi') \cap F$. Choose from the interval $[y_0 - \delta, y_0 + \delta]$ a point x_φ such that $h(x_\varphi) = \varphi$. Then $(h(x_\varphi), \psi) \in F$, and so, $(x_\varphi, \psi) \in F^*$. Of course, $\xi(x_\varphi) = \psi$, and so, $(x_\varphi, \psi) \in \mathcal{G}(\xi)$, what proves that $\mathcal{G}(\xi) \cap F^* \neq \emptyset$.

From (7), (8), (9) we infer that relation (6) is proved and, consequently, we may infer that the function h^* is not almost continuous. Observe that $h^* = (g \circ f)_{|[y_0-\delta, y_0+\delta]}$, which, according to Lemma 1, means that $g \circ f \notin \mathcal{A}$. The last observation contradicts to (10). The obtained contradiction proves that $f \in RAP(\mathcal{A})$.

Now, we suppose that $f \notin RAP(\mathcal{Q}_c)$. Then there exists a function $\mu \notin \mathcal{Q}_c$ such that $\mu \circ f \in \mathcal{Q}_c$. Let $w_o \in I$ be a point which is not a point of quasi-continuity of μ. This means that

(10) there exist numbers $\varepsilon > 0$ and $\eta > 0$ such that

$$\mu(V) \setminus (\mu(w_o) - \varepsilon, \mu(w_o) + \varepsilon) \neq \emptyset$$

for an arbitrary open set $V \subset (w_o - \eta, w_o + \eta)$.

Let s_0, z_0 be arbitrary real numbers such that $f_{|[s_0,z_0]}$ is a homeomorphism and $w_o \in \operatorname{Int}(f([s_0, z_0]))$.

Let $\lambda \in (0, \eta]$ be a real number such that

$$(w_o - \lambda, w_o + \lambda) \subset \operatorname{Int}(f([s_0, z_0]))$$

and let $t_0 \in [s_0, z_0]$ be a number such that $f(t_0) = w_o$. Since

$$f^{-1}_{|[s_0,z_0]}((w_o - \lambda, w_o + \lambda))$$

is an open set in $[s_0, z_0]$, containing t_0, then, according to the quasi - continuity of $\mu \circ f$ at t_0, there exists an open (on the real line) set $W \subset f^{-1}_{[s_0,z_0]}((w_o - \lambda, w_o + \lambda))$ such that $\mu(f(W)) \subset (\mu(w_o) - \varepsilon, \mu(w_o) + \varepsilon)$. Of course, $f(W)$ is an open set in I and $f(W) \subset (w_o - \lambda, w_o + \lambda)$, what contradicts (10).

The obtained contradiction ends the proof of the theorem.

From the above theorem and the well known Maximoff Theorem ([8], [9]) we can deduce (applying Lemma 4) the following corollary:

Corollary 1 *Let $f : I \longrightarrow I$ be a function of Baire class one. Then f is an almost continuous function if and only if there exists a function $h \in \widehat{H}$ such that $f \circ h$ is a derivative.*

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