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Continuity Points of a Multifunction in a Term of the Function "dist"

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In the article we shall characterize the set of points of continuity of a multifunction with values in a compact metric space. We make no restrictions for the values of a multifunction except the nonemptiness of them.

Let Y be a metric space, $z \in Y$ and $A \subset Y$. By $\operatorname{dist}(z, A)$ we understand the distance of the point z from the set A. By $\overline{A}$ we denote the closure of A, A' will denote the complement of A. We shall discuss multifunctions F defined on a topological Fréchet space X with nonempty values in a metric space Y.

A multifunction $F : X \longrightarrow Y$ is called upper - semicontinuous at $x_0 \in X$ iff for each open set U for which $\overline{F(x_0)} \subset U$ there exists an open neighbourhood V of x_0 such that $F(x) \subset U$ for every $x \in V$. The set of all points of the set X at which F is upper - semicontinuous will be denoted by $usc(F)$.

A multifunction $F : X \longrightarrow Y$ is called lower - semicontinuous at $x_0 \in X$ iff for each open set U for which $U \cup F(x_0) \neq \emptyset$ there exists an open neighbourhood V of x_0 such that $F(x) \cup U \neq \emptyset$ for every $x \in V$. The set of all points of the set X at which F is lower - semicontinuous will be denoted by $lsc(F)$.

Let z be a fixed point of Y. For a given multifunction $F : X \longrightarrow Y$ we define a function $f_z : X \longrightarrow \mathbb{R}$ in the following way:

$$f_z(x) = \operatorname{dist}(z, F(x)).$$

Theorem 1 *If a multifunction $F : X \longrightarrow Y$ is upper - semicontinuous at $x_0 \in X$, then the function f_z is lower - semicontinuous at x_0 for each $z \in Y$.*

Proof. Let $z \in Y$ and suppose that the function f_z is not lower - semicontinuous at x_0, i.e.

$$\liminf_{x \longrightarrow x_0} f_z(x) < f_z(x_0).$$

Then there exists a sequence $\{x_n\}$ convergent to x_0 and $\alpha \in \mathbb{R}$ such that

$$\lim_{n \longrightarrow \infty} f_z(x_n) = \alpha < f_z(x_0).$$

Let

$$U = \left\{ y \in Y : \operatorname{dist}(z, y) > \alpha + \frac{f_z(x_0)}{2} \right\}.$$

Then $\overline{F(x_0)} \subset U$ and $F(x_k) \not\subset U$ for almost all $k \in N$.

Corollary 1 *If a multifunction $F : X \longrightarrow Y$ is upper - semicontinuous then for every $z \in Y$ each of the function f_z is lower - semicontinuous.*

The analogous proof can be presented for the next theorem.

Theorem 2 *If a multifunction $F : X \longrightarrow Y$ is lower - semicontinuous at x_0, then f_z is upper - semicontinuous at x_0 for each $z \in Y$.*

Corollary 2 *If a multifunction $F : X \longrightarrow Y$ is lower - semicontinuous then for every $z \in Y$ each of the function f_z is upper - semicontinuous.*

We shall show, that, with the additional assumption for the space Y, the converse theorems are also valid.

Theorem 3 *If $F : X \longrightarrow Y$ is a multifunction, X is a metric space and Y – a compact metric space and for each $z \in Y$ the function $f_z : X \longrightarrow \mathbb{R}$ is lower - semicontinuous at a point $x_0 \in X$ then F is upper - semicontinuous at x_0.*

Proof. Suppose that F is not upper - semicontinuous at x_0. Then there exists an open set U such that $\overline{F(x_0)} \subset U$ for which for every neighbourhood V of the point x_0 there is $x \in V$ such that $F(x) \not\subset U$. In that way we define two sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n \in X$ and $y_n \in F(x_n) \setminus U$. Let $z \in Y$ be an accumulation point of the sequence $\{y_n\}$. Then $z \notin U$ and $\liminf_{x \longrightarrow x_0} f_z(x) = 0$, but $f_z(x_0) > 0$ – a contradiction.

Corollary 3 *Let X and Y be metric spaces, Y – a compact one. If $F : X \longrightarrow Y$ is a multifunction for which f_z is lower - semicontinuous for each $z \in Y$, then F is upper - semicontinuous*

In the analogous way we prove the next theorem:

Theorem 4 *Let X be a metric space, Y a compact metric space. If $F : X \longrightarrow Y$ is a multifunction for which f_z is upper - semicontinuous at x_0 for each $z \in Y$, then F is lower - semicontinuous at x_0.*

Proof. Suppose that F is not lower - semicontinuous at x_0. Then there is an open set U such that $F(x_0) \cap U \neq \emptyset$ and in each neighbourhood V of the point x_0 there is an $x \in X$ such that $F(x) \cap U = \emptyset$. Let $z \in F(x_0) \cap U$. Then $f_z(x_0) = 0$ and

$$\limsup_{x \longrightarrow x_0} f(x) \geq \operatorname{dist}(z, Y \setminus U) > 0.$$

The contradiction completes the proof.

Corollary 4 *Let X and Y be metric spaces, Y - a compact one. If for a multifunction $F : X \longrightarrow Y$ each of the functions f_z are upper - semicontinuous, then F is lower - semicontinuous.*

We shall give a similar characterization by the aid of a smaller family of functions f_z.

Theorem 5 *Let $F : X \longrightarrow Y$ be a multifunction from a topological space X to a metric space Y. Then the functions f_z are lower - semicontinuous at a point x_0 for each $z \in Y$ if and only if for some set Θ dense in Y the functions f_z are lower - semicontinuous at x_0 for each $z \in \Theta$.*

Proof. The necessity condition is obvious. Assume that for some z_0 the function f_{z_0} is not lower - semicontinuous at a point x_0. Let $\{z_n\}$ be such a sequence that $z_n \in \Theta, z_n \longrightarrow z_0$. Then there are α, β such that

$$\liminf_{x \longrightarrow x_0} f_{z_0}(x) < \alpha < \beta < f_{z_0}(x_0).$$

Notice, that for every $x \in X$

$$|f_{z_0}(x) - f_z(x)| \leq \operatorname{dist}(z_0, \{z\}).$$

Therefore one can choose such a sequence $\{x_n\}$ convergent to x_0 that

$$f_{z_n}(x_k) < \alpha \text{ and } f_{z_n}(x_0) > \beta$$

for almost all k-s, what is impossible in view of lower - semicontinuity of f_{z_n}.

Theorem 6 *Let X be a topological Fréchet space and Y a metric space. The functions f_z for $z \in Y$ are upper - semicontinuous at x_0 if and only if for each dense in Y subset Θ the functions f_z are upper - semicontinuous at x_0 for each $z \in \Theta$.*

Let us denote by $C(F)$ the set of all points of continuity of the multifunction F (i.e. $C(F) = usc(F) \cap lsc(F)$) and by $C(f)$ – the set of all points of continuity of the function f. Then the set $C(F)$ can be characterized in the following way:

$$C(F) = \bigcap_{z \in Y} C(f_z) = \bigcap_{z \in \Theta} C(f_z),$$

where Θ is a dense subset of Y.

Theorem 7 *Let X be a metric space, Y be a compact metric space. If $F : X \longrightarrow Y$ is any multifunction, then the set $C(F)$ is of the type G_δ.*

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