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On the rate of strong summability of Fourier-Chebyshev series

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1 Preliminaries.

Let us consider the system of Chebyshev polynomials

$$T_0(x) = 1, \quad T_n(x) = \cos(n \arccos x) \quad (n = 1, 2, \ldots),$$

which are ortogonal with the weight $\varrho(x) = (1-x^2)^{-\frac{1}{2}}$ on the interval $[-1, 1]$. Let f be a real-valued function continuous on $[-1, 1]$, and let

$$S[f](x) = \sum_{k=0}^{\infty} c_k T_k(x) \tag{1}$$

be its Fourier series with respect to the system $\{T_n(x)\}$, that is

$$c_0 = \frac{1}{\pi}\int_{-1}^{1} f(t)\varrho(t)\,dt, \quad c_k = \frac{2}{\pi}\int_{-1}^{1} f(t)T_k(t)\varrho(t)\,dt \quad (k = 1, 2, \ldots). \tag{2}$$

Denote by $S_n(x; f)$ the n-th partial sum and by $\sigma_n^\alpha(x) = \sigma_n^\alpha(x; f)$ the n-th Cesàro (C, α)-mean of order α of the series (1). Introduce the modulus of continuity of f defined by

$$\omega(\delta) = \omega(\delta, f) = \sup_{-\delta\le h\le\delta}\left\{\sup_{-1\le y, y+h\le 1} |f(y+h) - f(y)|\right\},$$

and the best approximation of f by polynomials P_n of the degree at most n given by

$$E_n(f) = \inf_{P_n} \left\{ \sup_{-1 \le x \le 1} |f(x) - P_n(x)| \right\}.$$

Take into account a regular summability method determined by a triangular matrix $\|\alpha_{nk}/A_n\|$ with $\alpha_{nk} \ge 0$ and $A_n = \sum_{k=0}^{n} \alpha_{nk}$. Write, for each non-negative integer n,

$$R_n(x;\alpha)_p = \left\{ \frac{1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} \left| \sigma_\nu^{\alpha-1}(x;f) - f(x) \right|^p \right\}^{1/p}, \quad p > 0. \tag{3}$$

The aim of this note is to estimate the quantity (3).

In considerations, the suitable positive constants independent of n, f and r are denoted by K_j $(j = 1, 2, \ldots)$, and

$$\gamma(\nu) = \min(2^\nu - 2, n), \quad \gamma_1(\nu) = 2^{\nu-1}, \quad \gamma_2(\nu) = \min(2^{\nu+1} - 2, n).$$

2 Auxiliary results

We start with an analogue of the Brudnyi and Gopengauz theorem (Th. 4, [1], p. 892).

Proposition. *Let*

$$F(z) = \sum_{k=0}^{\infty} c_k z^k, \quad z = r\,e^{ix}, \quad 0 < r < 1, \tag{4}$$

where c_k are defined by (2). Then

$$|F'(z)| \le K_1 \int_{1-r}^{1} \frac{\omega(t)}{t^2}\, dt \quad \textit{uniformly in } x \in [-\pi; \pi]. \tag{5}$$

Proof. An easy computation gives

$$F(z) = c_0 + \frac{2}{\pi} \sum_{k=1}^{\infty} r^k \int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} \cos(k \arccos t) e^{ikx}\, dt =$$

$$= \frac{2}{\pi} \int_0^\pi f(\cos y) \left\{ \frac{1}{2} + \sum_{k=1}^{\infty} r^k \cos ky \cos kx + i \sum_{k=1}^{\infty} r^k \cos ky \sin kx \right\} dy =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\cos y)\left\{P(r, y+x) + P(r, y-x)\right\} dy +$$
$$+ i\,\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\cos y)\left\{Q(r, y+x) - Q(r, y-x)\right\} dy =$$
$$= \frac{1}{2}\left\{u(r,-x) + u(r,x) - i\,v(r,-x) + i\,v(r,x)\right\},$$

where $u(r,x)$, $v(r,x)$ are the Poisson and the Poisson conjugate integrals of $f(\cos y)$ and $P(r,s)$, $Q(r,s)$ are their kernels, respectively. Thus

$$|F'(z)| = \left|\frac{\partial}{\partial x}F(z)\cdot\left(\frac{\partial z}{\partial x}\right)^{-1}\right| = \frac{1}{r}\left|\frac{\partial F(z)}{\partial x}\right| =$$
$$= \frac{1}{2r}\left|\frac{\partial(u(r,-x)+u(r,x))}{\partial x} - i\,\frac{\partial(v(r,-x)+v(r,x))}{\partial x}\right|.$$

Therefore, it is sufficient to show that the both terms of the right-hand side of the above relation are of the order

$$O\left(\int_{1-r}^{1} t^{-2}\omega(t)\,dt\right).$$

Since

$$\int_{-\pi}^{\pi}\frac{\partial}{\partial s}P(r,s)\,ds = 0,$$

we have

$$\frac{\partial u(r,\pm x)}{\partial x} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\cos y)\,\frac{\partial P(r, y\pm x)}{\partial x}\,dy =$$
$$= \pm\frac{1}{\pi}\int_{-\pi}^{\pi}\left\{f(\cos y) - f(\cos x)\right\}\frac{\partial P(r, y\pm x)}{\partial y}\,dy =$$
$$= \pm\frac{1}{\pi}\int_{-\pi}^{\pi}\left\{f(\cos(t\mp x)) - f(\cos x)\right\}\frac{\partial P(r,t)}{\partial t}\,dt.$$

By the mean-value theorem, for arbitrary real x, t and some $\theta\in(0,1)$,

$$|f(\cos(t+x)) - f(\cos x)| = |f(\cos x - t\sin(x+\theta t)) - f(\cos x)| \le$$
$$\le \sup_{|h|\le|t|}\left\{\sup_{-1\le y, y-h\le 1}|f(y-h) - f(y)|\right\} =$$
$$= \omega(|t|; f) \le (\pi+1)\,\omega\left(\frac{|t|}{\pi}; f\right).$$

Analogously,

$$|f(\cos(t-x))-f(\cos x)|\le(\pi+1)\,\omega\left(\frac{|t|}{\pi};f\right).$$

Hence

$$\begin{aligned}\frac{1}{r}\left|\frac{\partial u\,(r,\pm x)}{\partial x}\right| &\le \left(1+\frac{1}{\pi}\right)(1-r^2)\int_{-\pi}^{\pi}\omega\left(\frac{|t|}{\pi};f\right)\frac{|\sin t|}{(1-2r\cos t+r^2)^2}\,dt=\\ &= 2\left(1+\frac{1}{\pi}\right)(1-r^2)\int_0^{\pi}\omega\left(\frac{t}{\pi}\right)\frac{\sin t}{((1-r)^2+4r\sin^2(t/2))^2}\,dt\le\\ &\le 2\left(1+\frac{1}{\pi}\right)(1-r^2)\int_0^{\pi}\omega\left(\frac{t}{\pi}\right)\frac{t}{((1-r)^2+4r(t/\pi)^2)^2}\,dt=\\ &= 2\pi(\pi+1)(1-r^2)\int_0^1\frac{s\omega(s)}{((1-r)^2+4r\,s^2)^2}\,ds.\end{aligned}$$

Arguming further as in [5], pp. 150–151 we obtain

$$\frac{1}{r}\left|\frac{\partial u\,(r,\pm x)}{\partial x}\right|\le K_2\frac{\omega(1-r)}{1-r}$$

and this immediately implies

$$\frac{1}{r}\left|\frac{\partial v\,(r,\pm x)}{\partial x}\right|\le K_3\int_{1-r}^{1}t^{-2}\omega(t)\,dt.$$

Now the desired assertion is evident.

Remark 1. When $x=\arccos y$, estimate (5) holds uniformly in $y\in[-1,1]$.

Lemma 1. *If* $\alpha>\frac{1}{2}$, $k>0$ *and* $k(1-\alpha)<1$, *then*

$$\left\{\frac{1}{n+1}\sum_{\nu=n}^{2n}\left|\sigma_\nu^\alpha(x;f)-\sigma_\nu^{\alpha-1}(x;f)\right|^k\right\}^{1/k}\le K_3\frac{1}{n+1}\sum_{\nu=0}^{n}\omega\left(\frac{1}{\nu+1}\right)\tag{6}$$

uniformly in x *and* $n=0,1,2,\ldots$.

Proof. Let us consider the power series (4) and denote by $\tau_m^\alpha(y)$ its m-th (C,α)-means for $z = e^{i\arccos y}$. Easy calculation gives

$$\alpha A_m^\alpha(\tau_m^{\alpha-1}(y) - \tau_m^\alpha(y)) = \sum_{\nu=0}^{m} A_{m-\nu}^{\alpha-1} c_\nu e^{i\nu\arccos y};$$

whence, for $z = r^{i\varphi}$ $(0 < r < 1)$,

$$\alpha \sum_{m=0}^{\infty} A_m^\alpha(\tau_m^{\alpha-1}(y) - \tau_m^\alpha(y))\, z^m = z\, e^{i\arccos y} F'(z\, e^{i\arccos y})\,(1-z)^{-\alpha}.$$

Further, taking a number $p \in (1,2)$ such that $p\alpha > 1$, by Hausdorff-Young inequality with $q = \frac{p}{p-1}$, we get

$$\left\{\sum_{m=0}^{\infty} m^{\alpha q} \left|\tau_m^{\alpha-1}(y) - \tau_m^\alpha(y)\right|^q r^{mq}\right\}^{\frac{1}{q}} \le K_4 \left\{ r^p \int_{-\pi}^{\pi} \frac{|F'(re^{i\varphi}e^{i\arccos y})|^p}{|1 - re^{i\varphi}|^{\alpha p}}\, d\varphi \right\}^{\frac{1}{p}}.$$

In view of Proposition, the right-hand side of the above inequality does not exceed

$$K_5 \left\{ r^p \int_{-\pi}^{\pi} \left(\int_{1-r}^{1} t^{-2}\omega(t)\, dt\right)^p \frac{d\varphi}{((1-r)^2 + \varphi^2)^{\alpha p/2}} \right\}^{1/p}.$$

Setting $1 - r = 1/(2n+1)$ and taking the real part we obtain our thesis (see [5], pp. 152–153).

Lemma 2. *Suppose that for each $q > 1$, the condition*

$$\left\{\frac{1}{2^\nu} \sum_{k=\gamma_1(\nu)}^{\gamma(\nu)} (\alpha_{nk})^q \right\}^{1/q} \le K_6\,(2^{-\nu}) \sum_{k=\gamma_1(\nu)}^{\gamma(\nu)} \alpha_{nk} \tag{7}$$

holds for $\nu = 1, 2, \ldots, \nu_n$, whenever $2^{\nu_n - 1} \le n + 1 < 2^{\nu_n - 1}$. Then, for $p \in (0, \infty)$,

$$\left\{ \left(\sum_{k=\gamma_1(\nu)}^{\gamma(\nu)} \alpha_{nk}\right)^{-1} \sum_{k=\gamma_1(\nu)}^{\gamma(\nu)} \alpha_{nk}\, |S_k(x; f) - f(x)|^p \right\}^{1/p} \le K_7\, E_{\gamma_1(\nu)}(f).$$

Proof. Proceeding analogously as in [4] and using Theorem 3 from [6] we get the desired result.

3 Main results

Now we present two approximation theorems which are analogues of some Leindler's results ([2], [3], [4]).

Theorem 1. *Under the assumptions of Lemma 2, we have*

$$R_n(x;1)_p \le \left\{ \frac{K_8}{A_n} \sum_{\nu=1}^{\nu_n} \left(\sum_{k=\gamma_1(\nu)}^{\gamma(\nu)} \alpha_{nk} \right) \left(E_{\gamma_1(\nu)}(f) \right)^p \right\}^{1/p} \quad \textit{for } n = 0, 1, 2, \ldots .$$

Proof. The above estimate is a consequence of Lemma 2.

In particular if $p = 1$, $\alpha_{nk} = A_{n-k}^{\beta-1}$, $\beta > 0$ the condition (7) holds. Thus we obtain

Corollary. *Under the assumption $\beta > 0$,*

$$\frac{1}{A_n^\beta} \sum_{k=0}^{n} A_{n-k}^{\beta-1} \left| S_k(x;f) - f(x) \right| \le K_9 \frac{1}{n+1} \sum_{k=0}^{n} E_k(f) \quad \textit{for } n = 0, 1, 2, \ldots .$$

Passing to the more general means (3), let us put

$$\varphi(\nu) = \frac{1}{\nu} \sum_{k=0}^{\nu} \omega\left(\frac{1}{k+1} \right).$$

Theorem 2. *Suppose that $\alpha > 1/2$, $p > 0$, $p(1-\alpha) < 1$ and that*

$$\sum_{l=0}^{m} \left\{ \sum_{\nu=2^l-1}^{\gamma_2(l)} \frac{\alpha_{n\nu}^q (\varphi(\nu))^{qp}}{(\nu+1)^{1-q}} \right\}^{1/q} \le K_{10} A_n \frac{1}{n+1} \sum_{\nu=0}^{n} (\varphi(\nu))^p \tag{8}$$

for each $q > 1$, whenever $2^m \le n+1 < 2^{m+1} - 1$. Then

$$\max_{-1 \le x \le 1} R_n(x;\alpha)_p \le K_{11} \left\{ \frac{1}{n+1} \sum_{\nu=0}^{n} (\varphi(\nu))^p \right\}^{1/p} \quad \textit{for } n = 0, 1, 2, \ldots . \tag{9}$$

Proof. Clearly,

$$(R_n(x;\alpha)_p)^p \le$$

$$\le 2^p \left\{ \frac{1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} \left|\sigma_\nu^{\alpha-1}(x) - \sigma_\nu^\alpha(x)\right|^p + \frac{1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} \left|\sigma_\nu^\alpha(x) - f(x)\right|^p \right\} =$$

$$= 2^p \left\{ \sum\nolimits_1 + \sum\nolimits_2 \right\}.$$

Choose two numbers $p', q' > 1$ such that $pp'(1-\alpha) < 1$ and $\frac{1}{p'} + \frac{1}{q'} = 1$. In view of estimate (6) and condition (8),

$$\sum\nolimits_1 = \frac{1}{A_n} \sum_{l=0}^{m} \left(\sum_{\nu=2^l-1}^{\gamma_2(l)} \alpha_{n\nu} \left|\sigma_\nu^{\alpha-1}(x) - \sigma_\nu^\alpha(x)\right|^p \right) \le$$

$$\le \frac{1}{A_n} \sum_{l=0}^{m} B_{n,l} \left(\sum_{\nu=2^l-1}^{\gamma_2(l)} (\nu+1)^{-1} \left|\sigma_\nu^{\alpha-1}(x) - \sigma_\nu^\alpha(x)\right|^{pp'} \right)^{\frac{1}{p'}} \le$$

$$\le \frac{1}{A_n} \sum_{l=0}^{m} B_{n,l} \left(\frac{1}{2^l} \sum_{\nu=2^l-1}^{\gamma_2(l)} \left|\sigma_\nu^{\alpha-1}(x) - \sigma_\nu^\alpha(x)\right|^{pp'} \right)^{\frac{1}{p'}} \le$$

$$\le \frac{K_3^p}{A_n} \sum_{l=0}^{m} B_{n,l} (\varphi(2^l - 1))^p \le \frac{(2K_3)^p}{A_n} \sum_{l=0}^{m} B_{n,l} (\varphi(2^{l+1} - 2))^p \le$$

$$\le \frac{(2K_3)^p}{A_n} \sum_{l=0}^{m} \left(\sum_{\nu=2^l-1}^{\gamma_2(l)} \frac{\alpha_{n\nu}^{q'} (\varphi(\nu))^{q'p}}{(\nu+1)^{1-q'}} \right)^{\frac{1}{q'}} \le (2K_3)^p K_{10} \frac{1}{n+1} \sum_{\nu=0}^{n} (\varphi(\nu))^p,$$

where $B_{n,l} = \left(\sum_{\nu=2^l-1}^{\gamma_2(l)} \alpha_{n\nu}^{q'} (\nu+1)^{q'-1} \right)^{\frac{1}{q'}}$.

By Corollary, for $s \in (1, \infty)$,

$$\sum\nolimits_2 \le \frac{K_{12}}{A_n} \sum_{\nu=0}^{m} \alpha_{n\nu} (\varphi(\nu))^p \le \frac{K_{12}}{A_n} \sum_{l=0}^{m} 2^l \left\{ \frac{1}{2^l} \sum_{\nu=2^l-1}^{\gamma_2(l)} \alpha_{n\nu} (\varphi(\nu))^p \right\} \le$$

$$\le \frac{K_{12}}{A_n} \sum_{l=0}^{m} 2^l \left\{ \frac{1}{2^l} \sum_{\nu=2^l-1}^{\gamma_2(l)} \alpha_{n\nu}^s (\varphi(\nu))^{ps} \right\}^{\frac{1}{s}} =$$

$$= \frac{K_{12}}{A_n} \sum_{l=0}^{m} \left\{ 2^{l(s-1)} \sum_{\nu=2^l-1}^{\gamma_2(l)} \alpha_{n\nu}^s (\varphi(\nu))^{ps} \right\}^{\frac{1}{s}} \leq$$

$$\leq \frac{K_{12}}{A_n} \sum_{l=0}^{m} \left\{ \sum_{\nu=2^l-1}^{\gamma_2(l)} \frac{\alpha_{n\nu}^s (\varphi(\nu))^{ps}}{(\nu+1)^{1-s}} \right\}^{\frac{1}{s}} \leq K_{10} \cdot K_{12} \cdot \frac{1}{n+1} \sum_{\nu=0}^{n} (\varphi(\nu))^p.$$

Summing up these estimates we obtain (9).

Remark 2. Condition (8) is satisfied in the special case when $\alpha_{nk} = A_{n-k}^{\beta-1}$ and $\beta > 0$.

Indeed, observing that

$$\varphi(k) \leq \varphi(k') \quad \text{when } k \geq k'$$

and

$$\varphi(2^l - 1) \leq 2\varphi(k) \quad \text{when } 2^l - 1 \leq k \leq 2^{l+1} - 2$$

and also the fact that $\alpha_{nk} = A_{n-k}^{\beta-1}$ $(\beta > 0)$ satisfy (7), we obtain

$$\sum_{l=0}^{m} \left\{ \sum_{\nu=2^l-1}^{\gamma_2(l)} \frac{\left(A_{n-\nu}^{\beta-1}\right)^q (\varphi(\nu))^{qp}}{(\nu+1)^{1-q}} \right\}^{1/q} \leq$$

$$\leq 2 \sum_{l=0}^{m} \left\{ (\varphi(2^l-1))^p \cdot 2^l \left(\frac{1}{2^l} \sum_{\nu=2^l-1}^{\gamma_2(l)} \left(A_{n-\nu}^{\beta-1}\right)^q \right)^{1/q} \right\} \leq$$

$$\leq 2 \cdot K_{13} \sum_{\nu=0}^{m} \left((\varphi(2^l-1))^p \sum_{\nu=2^l-1}^{\gamma_2(l)} A_{n-\nu}^{\beta-1} \right) \leq$$

$$\leq 4 \cdot K_{13} \sum_{\nu=0}^{m} \left(\sum_{\nu=2^l-1}^{\gamma_2(l)} A_{n-\nu}^{\beta-1} (\varphi(\nu))^p \right) =$$

$$= 4 \cdot K_{13} \sum_{\nu=0}^{n} A_{n-\nu}^{\beta-1} (\varphi(\nu))^p =$$

$$= 4 \cdot K_{13} \left(\sum_{\nu=0}^{\left[\frac{n}{2}\right]} + \sum_{\nu=\left[\frac{n}{2}\right]+1}^{n} \right) A_{n-\nu}^{\beta-1} (\varphi(\nu))^p \leq$$

$$\leq K_{14}\left((n+1)^{\beta-1}\sum_{\nu=0}^{\left[\frac{n}{2}\right]}(\varphi(\nu))^p+\left(\varphi\left(\left[\frac{n}{2}\right]+1\right)\right)^p\sum_{\nu=\left[\frac{n}{2}\right]+1}^{n}A_{n-\nu}^{\beta-1}\right)\leq$$

$$\leq K_{15}(n+1)^{\beta}\left(\frac{1}{n+1}\sum_{\nu=0}^{n}(\varphi(\nu))^p+(\varphi(n))^p\right).$$

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Streszczenie

O rzędzie mocnej sumowalności szeregów Fouriera-Czebyszewa

Uzyskano oszacowania uogólnionych mocnych dewiacji sum częściowych i (C,α)-średnich szeregów Fouriera-Czebyszewa funkcji ciągłych od tych funkcji. Błędy oszacowań wyrażone zostały w terminach najlepszych przybliżeń i modułów ciągłości. Udowodnione twierdzenia stanowią uogólnione odpowiedniki wyników L. Leindlera.

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