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On asymmetry of multifunctions

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A multifunction $F : X \longrightarrow Y$ is a correspondence which assigns to each point in a set X at least one point in a set Y. If X and Y are topological spaces, $F : X \longrightarrow Y$ is said to be closed at a point $x_0 \in X$ if for each point $y_0 \in F(x_0)$ there exist two open neighbourhoods $V(x_0)$ and $U(y_0)$ of the points x_0 and y_0, respectively, such that the following implication holds

$$x \in V(x_0) \Longrightarrow F(x_0) \cap U(y_0) = \emptyset.$$

This notion was investigated e.g. in [11] under the name cofinal continuity. If F is cofinally continuous at each point $x \in X$, it is briefly called cofinally continuous (or sometimes p-usc). F is cofinally continuous if and only if its graph

$$\mathrm{Gr}F = \{(x, y) : y \in F(x)\} \subseteq X \times Y \tag{1}$$

is closed in the product space $X \times Y$. It is easy to see that F is cofinally continuous if and only if its inverse multifunction $F^{-1} : Y \longrightarrow X$, $F^{-1}(y) = \{x \in X : y \in F(x)\}$ is cofinally continuous. If $F : X \longrightarrow Y$ is cofinally continuous at a point x_0, then the value $F(x_0)$ is closed in Y. For compact range spaces Y, $F : X \longrightarrow Y$ is cofinally continuous if and only if F is upper semicontinuous and has closed values. Let us recall, that a multifunction $F : X \longrightarrow Y$ is upper semicontinuous if the inverse image

$$F^{-}(D) = \{x \in X : F(x) \cap D \neq \emptyset\} \tag{2}$$

of each set D closed in Y, is closed in X. For more informations concerning to the existing various kinds of continuity of multifunctions see [10], [11], [9]. The notion of cluster set for multifunctions is introduced in [14], [5] and [8] as follows. We say that y is an element of the cluster set of $F : X \longrightarrow Y$ at x, denoted $C(F, x)$, if there exists a net $x_d \in X$, $d \in D$ and a net $y_d \in F(x_d)$ such that x_d is MS-convergent to x and y_d is MS-convergent to y. D is here an apropriate directed set. Thus the cluster set may be treated as a formalization of the set of "limit points" of our multifunction F.

Denoting by CL the closure operator in Y, we have

$$C(F,x) = \bigcap\{\mathrm{CL}\,F(U) : U \in N(x)\},$$

where $N(x)$ denotes the filterbase of neighbourhoods at x, and the image $F(U)$ is defined for multifunctions as follows

$$F(U) = \bigcup\{F(x) : x \in U\}.$$

In [5] the following theorem is essentially proved.

Theorem 0 [5]. *The following conditions are equivalent for multifunction $F : X \longrightarrow Y$*

(i) *F has closed graph (1) i.e. F is cofinally continuous*

(ii) $F(x) = \mathrm{Pr}_Y\left(\{x\} \times Y \cap \mathrm{CL}(\mathrm{Gr}F)\right), \quad x \in X$

(iii) $C(F,x) = F(x), \quad x \in X.$

An inspection of the proof gives also a local version of the above theorem: F is cofinally continuous at x if and only if $F(x) = C(F,x)$. In a case if F is single valued the cofinal continuity reduces to usual continuity provided the range space is compact and then we obtain from theorem 0 the result of Weston [13]. The reader can easily construct an example where $\mathrm{card}F(x) = 1$, i.e. $F(x) = \{f(x)\}$, $C(f,x)$ is a singleton yet f is not continuous using a non-compact range space Y. At present we define a relation on topologies analogous to one defined by Ulysses Hunter in [4]. Applying cluster set techniques we obtain some information on the set of cofinal discontinuities of an arbitrary multifunction. This seems to be of some interest in connection with the result of [6], where the conditions on Borel measurability of multifunctions of two variables are formulated and also in connection with the result of [1], where in terms of cluster sets J. Ceder characterised multifunctions possessing a selector with the Darboux property.

A collection M of subsets of X is called a σ-ideal of sets in X if

(a) $A \in M$ and $B \subseteq A$ implies that $B \in M$,

(b) $A_n \in M$, $n = 1, 2, 3, \ldots$, implies that $\bigcup_{n=1}^{\infty} A_n \in M$,

(c) $X \notin M$.

Two topologies T and S on X are said to be related modulo M, denoted T *rel* $S \bmod M$, if for any subset A of X, the $T-$ and $S--$ closure of A differ by a set in M

$$\mathrm{CL}_T(A) \triangle \mathrm{CL}_S(A) \in M, \ A \triangle B = (A \setminus B) \cup (B \setminus A).$$

Theorem 1 *Let M be a σ-ideal of subsets of a bitopological space (X, T, S) and T rel $S \bmod M$. If $F : X \longrightarrow Y$ is an arbitrary multifunction from the space X into the second countable space Y, then*

$$C_T(F, x) = C_S(F, x) \quad \textit{for every } x \in X \setminus A \quad \textit{for some} \quad A \in M$$

Proof. Let $\{U_n : n = 1, 2, 3, \ldots\}$ be a countable open basis for Y. Observe that

$$\mathrm{Asym} F = \{x \in X : \ C_T(F, x) \neq C_S(F, x)\} = E \cup D,$$

where

$$E = \{x \in X : \ C_T(F, x) \not\subseteq C_S(F, x)\}$$

and

$$D = \{x \in X : \ C_S(F, x) \not\subseteq C_T(F, x)\}.$$

To show that $E \in M$, put for $n = 1, 2, \ldots$

$$E_n = \{x \in X : \ x \in \mathrm{CL}_T F^{-}(U_n) \text{ and } x \notin \mathrm{CL}_S F^{-}(U_n)\}$$

where the big inverse image is defined by formula (2). Note that in compliance with the assumption that $T\ \ rel\ \ S \bmod M$, each E_n belongs to M and hence, by virtue of (b) also $\bigcup_{n=1}^{\infty} E_n$ is in M. We claim that E is included in this union $\bigcup_{n=1}^{\infty} E_n$. Indeed, if x is in E then there exists an y in Y such that for every $V \in N(y)$ and for every $G \in N(x)$

$$F(G) \cap V \neq \emptyset \quad \text{i.e. } x \in \mathrm{CL}_T F^-(V),$$

but $F^-(N(y))$ does not accumulate at x with respect to S, that means there is a V_1 in $N(y)$ and G_1 in $N_S(x)$ such that

$$F(G_1) \cap V_1 = \emptyset \quad i.e. \quad x \notin \mathrm{CL}_S F^-(V_1).$$

Since $\{U_n : n = 1, 2, 3, \ldots\}$ is a basis for Y, there exists some positive integer k such that $y \in U_k \subseteq V_1$. We conclude that x is in E_k and hence we have the claim. Thus $E \in M$ and a similar argument shows that $D \in M$, completing the proof.

Remark 1 In case of single valued functions this theorem reduces to a part of theorem 1 on p.78 in [12]. Note that Świątkowski wrongly assumed the separability of Y instead of second axiom of countability in his theorem.

Theorem 2 *Let $F : X \longrightarrow Y$ be an arbitrary multifunction from X into the second countable Hausdorff space Y. Let M be a σ-ideal of subsets of X with T and S two topologies on X related modulo M.*

(a) *If F is cofinally continuous with respect to T, then the set of cofinal discontinuities of F with respect to S is an element of M.*

(b) *If $F : X \longrightarrow Y$ is either cofinally continuous with respect to S or with respect to T at every point of X, then the set of cofinal discontinuities with respect to the intersection $T \cap S$ is an element of M.*

Proof. By theorem 0 we have $F(x) = C_T(F, x)$ for all $x \in X$. Applying theorem 1 we find a set $A \in M$ such that $C_S(F, x) = C_T(F, x)$ for $x \in X \setminus A$. Therefore $F(x) = C_S(F, x)$ for $x \in X \setminus A$ and, taking into account once again the local version of theorem 0 we deduce that F is cofinally continuous with respect to S at all points $x \in X \setminus A$.

To prove part (b) of Theorem 2, first observe that

$$C_T(F,x) \cap C_S(F,x) \subseteq C_{T\cap S}(F,x) \subseteq C_T(F,x) \cup C_S(F,x).$$

Now, there exists an subset A belonging to M such that for $x \in X \setminus A$ we have that $C_T(F,x) = C_S(F,x)$. Since our multifunction F is cofinally continuous either with respect to T or with respect to S, we have that

$$C_T(F,x) = C_S(F,x) = F(x) \subseteq Y \text{ iff } x \in X \setminus A.$$

Hence, if $x \in X \setminus A$ we have $C_{T\cap S}(F,x) = F(x)$ and consequently F is cofinally continuous at $x \in X \setminus A$ with respect to intersection of topologies $T \cap S$. We conclude that the set of cofinal discontinuities of F with respect to $T \cap S$ is a subset of A and hence an element of M. The proof is complete.

Remark 2 This theorem even in single-valued case is stronger than the corresponding one in [3] (th. 3.5). Namely Hamlett dealt with the topology generated by the union $T \cup S$ of original topologies instead if its intersection.

In case where Y is compact we obtain the following corollary.

Corollary 1 *Let $F : X \longrightarrow Y$ be an upper semicontinuous multifunctions with compact range Y and closed values. If S is another topology on X which is related to the initial topology T modulo M, then the set of points at which F fails to be upper semicontinuous with respect to S is an element of M. Moreover if at each point $x \in X$, F is either upper semicontinuous with respect to T or upper semicontinuous with respect to S, then it is upper semicontinuous with respect to both topologies simultaneously, except of a set of points belonging to M.*

An important consequence of Theorem 2 in case where X is the real line is the following:

Theorem 3 *Let $F : X \longrightarrow Y$ be an arbitrary multifunction from the real line into a second countable Hausdorff space. If F is either cofinally continuous from the right or cofinally continuous from the left at every* *real number $x \in X$, then F has at most countably many points, at which it fails to be cofinally continuous with respect to the Euclidean topology on X.*

Proof. Let M be the σ–ideal of countable sets of the real line X. If R is the topology on X generated by $\{[x, x+r): \ x \in X\ ,\ r > 0\}$ and L is the topology on X generated by $\{(x-r, x]: \ x \in X\ ,\ r > 0\}$ then it is easy to show that L is related to R modulo M (see e.g. Hunter [4], Example 1). Observe that $L \cap R$ is the usual Euclidean topology. It suffices to apply Theorem 2 to obtain the thesis.

In case when Y is compact and F has closed values, a similar theorem is valid for upper semicontinuity in place of cofinal continuity.

If Q is a subset of X then we can define the cluster set of $F: X \longrightarrow Y$ at $x \in X$ relative to Q as follows (cf. [8], [5]):

$$C(F, x, Q) = \bigcap \mathrm{CL} F(U \cap Q),$$

where the intersection is taken over all neighbourhoods $U \in N(x)$ of x in X with $F(U \cap Q) = \bigcup\{F(t): \ t \in U \cap Q\}$.

From this definition it is clear that $C(F, x, Q)$ is always closed (but possibly empty) subset of Y. In the theory of multivalued functions of complex variable esspecially important are radial, angular and curvilinear cluster sets obtained from (15) by suitably choosen Q. In order to generalize the Theorem 3 to higher dimensions we must define a notion of a point of asymmetry of a multifunction $F: X \longrightarrow Y$ where X stands for an n–dimensional Euclidean space. Let in this space, besides the natural topology, another topology T be distinguished.

We will denote by $\mathrm{CL}_T A$, $\mathrm{Der}_T A$ etc. the closure of the set $A \subseteq X$, the set if its accumulation points, etc. with respect to the topology T. When the topology T coincides with the natural topology, we omit this additional notation.

A point $x \in X$ will be called a T–assymmetry point of a multifunction $F: X \longrightarrow Y$ if there exists an $(n-1)$–dimensional hyperplane H passing through x such that

$$C_T(F, x, X^+) \neq C_T(F, x, X^-),$$

where X^+ and X^- denote the halfspaces being the components of the set $X \setminus H$. Observe that Theorem 3 on countability of the set of asymmetry points of a multifunction of one real variable cannot be carried over to multifunctions of many variables, as the following example shows

$$R^2 \ni (x,y) \longmapsto F(x,y) = \begin{cases} [0,7] & \text{for } x \leq 0 \\ [5,13] & \text{for } x > 0 \end{cases} \tag{17}$$

However, in this case the σ–ideal of countable sets may be replaced by the σ–ideal of meager sets

Theorem 4 *Let $F : X \longrightarrow Y$ be a multifunction defined on an n–dimensional Euclidean space X and taking his values in a second countable space Y. The set of assymetry points of F is of the first category.*

Proof. Acting as in the proof of Theorem 2 and 1 it suffices to prove that the sets

$$\mathrm{Asym}B = \mathrm{Der}(B \cap X^+) \triangle \mathrm{Der}(B \cap X^-),$$

for $B = F^-(U_n)$, $n = 1, 2, \ldots$ are nowhere dense. Namely let $B \subseteq X$ and let K be an arbitrary ball in the space X. We will show that there exists a ball $K_1 \subseteq K$ disjoint with the set Asym$B$. First, if K is disjoint with Asym$B$ it suffices to put $K_1 = K$. So let $x \in K \cap$ Asym$B$. Consequently there exists a decomposition $X = X^+ \cup H \cup X^-$ corresponding to the point x and such that $x \in$ Asym$B$. Without loss of generality we may assume that $x \notin \mathrm{Der}(B \cap X^+)$. Then there exists a ball K_0 centered at x such that $K_0 \cap X^+ \cap B = \emptyset$. Now let K_1 be a ball contained in $K_0 \cap X^+$. The set $K_1 \cap B \subseteq K_0 \cap X^+ \cap B$ must be empty which implies that the ball K_1 does not contain any accumulation point of the set B and consequently any of its asymmetry points. That ends the proof.

Remark 3 This proof is almost idenitcal with the proof of Theorem 4 in [12], only the meaning of the set B is different. Note that a formula (7) in Theorem 3 in cited Świątkowski's paper concerns the points of T–symmetry, but not T–asymmetry, as it is errorously stated.

Repeating the proofs of Theorem 5 and 6 from [12] with the obvious changes we can obtain a characterization of T–asymmetry points where

T is density topology, with respect to the ordinary differentiation basis. Such points will be called ordinary approximative asymmetry points.

In fact, we have

Theorem 5 *The ordinary approximative asymmetry points of a multifunction $F : X \longrightarrow Y$ defined on a finite dimensional Euclidean space X form a set of the first category. Moreover, the Lebesgue measure of the set of ordinary approximative asymmetry points of F is equal to zero.*

The σ-porosity of the set of approximative symmetry points, in the spirit of [15], as well as asymmetry points with respect to another fine topologies (Hashimoto topology e.g.) will be investigated in a later paper.

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