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ON CARATHEODORY'S SELECTORS FOR MULTIFUNCTIONS WITH VALUES IN S-CONTRACTIBLE SPACES

1. Introduction

Let X and Y be topological spaces. A multifunction $F:X \multimap Y$ is a function whose value $F(x)$ for $x \in X$ is a nonempty subset of Y. We say F is lower semicontinuous if $F^{-}(U) := \{x \in X\ F(x) \cap U \neq \emptyset\}$ is open in X whenever U is open in Y (cf. [10], [3]). This notion was introduced in 1932 independently by Kuratowski and Bougliand. If $F(x) = \{g(x)\}$ for a singlevalued function $g : X \to Y$, then lower semicontinuity of F reduces simply to the continuity of g.
Recall that a function $f: X \to Y$ is said to be a selector for F, if $f(x) \in F(x)$ for each $x \in X$.
A quadruple (X,M,m,T) is said to be a topological measure space iff (X,M,m) is a measure space and T is a topology on X with $T \subset M$.
Let (X,T) be a Hausdorff topological space and $B(X)$ the smallest tribe containing T. A positive Radon measure m on X (cf. [CV], p.62) is a positive measure $m: B(X) \to [0,\infty]$ such that:
(A) For each $x \in X$ there exists an open neighbourhood of x of finite measure
(B) For each Borel set $A \in B(X)$

$$m(A) = \sup\{m(K):\ K \subset A, K\text{-compact}\}$$

For a tribe M and a measure $m : M \to [0,\infty]$ we denote by m^* the outer measure induced by M. Then M_m denotes the completion $\{A \div N : A \in M \text{ and } m^*(N) = 0\}$.
In theorem 1 $\hat{M}$ denotes the intersection of all completions M_m where m runs through all positive bounded measures m on M. The sets belonging to $\hat{M}$ are called universally

measurable (cf. [CV] p.73)

A multifunction $F : X \to Y$ from a topological space with a positive Radon measure m into the nonempty subsets of a topological space, will be called measurable, if $F^-(U)$ is m-measurable for each open set U in Y.

If $F : X \times Y \to Z$ is a multifunction defined on a product of a topological measure space X and a topological space Y, taking values in a topological space Z, such that all X-sections $Y \ni y \mapsto F_x(y) := F(x,y) \subset Z$, $x \in X$ are lower semicontinuous and Y-sections $X \in x \mapsto F^y(x) := F(x,y) \subset Z$; $y \in Y$ are measurable, we may ask, under what conditions F admits a Caratheodory selector $f : X \times Y \to Z$, i.e. a function such that

a) $f(x,y) \in F(x,y)$ for all $(x,y) \in X \times Y$

b) $f_x : Y \to Z$ is continuous for all $x \in X$

c) $f^y : X \to Z$ is measurable for all $y \in Y$ (cf. [5])

There are many papers devoted to this problem see([19], [I], [K] [7], [8], [9], [11], [17]) The purpose of the present article is to generalize Castaing's result [8] onto the case of multifunctions taking values in S-comtractible spaces uniformly of of type O for balls.

2. Scorza-Dragoni property of F

Lemma 0. Let Y be a topological space, Z a separable metric space and let $F: Y \to Z$ be a multifunction. Then the statements:

(i) F is lower semicontinuous

(ii) $y \mapsto g_z(y) = \text{dist}[z, F(y)]$ is an upper semicontinuous single-values function of y for each z belonging to some countable dense subset of Z, are equivalent.

P r o o f. Let $K(z,r) = \{z_1 : d(z_1, z) < r\}$ denote a open ball in Z. F is lower semicontinuous iff $F^-[K(z,r)]$ is open in Y for each z belonging to a dense subset of Z and each $r > 0$. On the other hand g_z is upper semicontinuous in y iff $\{y : \text{dist}[z, F(y)] < r\}$ is open in Y for each $0 < r \leq +\infty$. But

$$F^-[K(z,r)] = \{y : F(y) \cap K(z,r) \neq \emptyset\} = \{y : d[z, F(y)] < r\}$$

It follows that (i)$\Leftrightarrow$(ii).

Lemma 1. Let X be a topological Hausdorff measure space with a positive, finite Radon measure m, and let Y be a Polish space. Let $f : X \times Y \to R$ be a real function such that:

(a) f is $M_m \otimes B(Y)$- measurable, where M_m denote the σ-algebra of all m-measurable subsets of X and $B(Y)$ denote the σ-algebra of all Borel subsets of Y.

(b) $f_x : Y \to R$ is upper semicontinuous (as a single-valued function) on Y for all $x \in X$.

Then there exists a decreasing sequence (f_n) of real functions defined on the product $X \times Y$ satisfying the following conditions :

(1) $f_n(\cdot, y)$ is m-measurable on X for each fixed $y \in Y$

(2) $f_n(x, \cdot)$ is continuous on Y for each fixed $x \in X$

(3) $\inf\{f_n : n \in N\} = f$.

Moreover, f has the following property: for each $\varepsilon > 0$, there exists some compact subset $K_\varepsilon \subset X$ such that $m(X \setminus K_\varepsilon) \le \varepsilon$ and that the restriction $f|K_\varepsilon \times Y$ is upper semicontinuous.

The following two theorems are indispensable in the proof of the lemma 1 :

Theorem 1 ([CV], lemma III. 39, p.86). Let (X,M) be a measurable space, Y a Souslin space (i.e. a continuous image of a Polish one), $F : X \times Y \to R$ a $M \otimes B(Y)$ -measurable function and $G: X \to Y$ a multifunction whose graph $Gr\, G = \{(x,y) \in X \times Y : y \in G(x)\}$ belongs to $M \otimes B(Y)$. Then $g(x) = \sup\{f(x,y) : y \in G(x)\}$ is a $\hat{M}$-measurable function of x.

Theorem 2. ([6],[4]). Let X be a compact topological space with a positive Random measure m and let Y be a Polish space. Suppose that all Y-sections of some function $h : X \times Y \to R$ are m-measurable on X, and all X-sections of h are continuous on Y. Then h has the Scorza-Dragoni property, viz. for each $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset K$ that $m(X \quad K_\varepsilon) \le \varepsilon$ and the restriction $h|K_\varepsilon \times Y$ is continuous on $K_\varepsilon \times Y$.

For the connection between the Scorza-Dragoni property and

Caratheodory property (see [16]).

Remark 1. Observe, that in theorem 3, X need only be assumed Hausdorff topological measure space with a positive finite Random measure m.

In fact, by virtue of (B) there exists a compact subset $K_1 \subset X$ such that $m(X \setminus K_1) \leqslant \varepsilon/2$. Then applying theorem 3 to K_1 we obtain $K_2 \subset K_1 \subset X$ for which $m(K_1 \setminus K_2) \leqslant \varepsilon/2$ and $h \mid K_2 \times Y$ is continuous. Obviously $m(X \setminus K_2) = m(X \setminus K_1) + m(K_1 \setminus K_2) \leqslant \varepsilon$.

PROOF OF THE LEMMA 1 :

Since all notions under considerations are invariant under homeomorphism $u \mapsto h(u) = \operatorname{arc\,tg}(u) - \pi/2$ and under its inverse h^{-1}, hence we may assume without any loss of generality that $f(x,y) \leqslant 0$ for all $(x,y) \in X \times Y$. For each $(x,y) \in X \times Y$ and each positive integer n, put

$$f_n(x,y) := \sup \{ f(x,y_1) - n\, d(y,y_1) \; : \; y_1 \in Y \} .$$

Obviously $f_{n+1} \leqslant f_n \leqslant 0$ for each n.

Further each function $f_n(\cdot,y)$ is measurable by virtue of theorem 1 invoked for $G(x) \equiv Y$.

Putting $y = y_1$ in formula defining f_n, we have

$f_n(x,y) \geqslant f(x,y)$ for each n and each $(x,y) \in X \times Y$.

On the other hand, applying the triangle inequality, we obtain

$$f_n(x,y_1) \geqslant \sup_{y \in Y} \{ f(x,y_1) - n\, d(y_1,y_2) - n\, d(y_2,y) \} =$$

$$= f_n(x,y_2) - n\, d(y_1,y_2) \quad \text{for each } n \text{ and each } (y_1,y_2) \in Y \times Y.$$

Hence

$$f_n(x,y_1) - f_n(x,y_2) \geqslant - n\, d(y_1,y_2)$$

and changing the role of y_1 and y_2 by a symmetric argument we obtain jointly

$$|f_n(x,y_1) - f_n(x,y_2)| \leqslant n\, d(y_1,y_2)$$

so that $f_n(x,\cdot)$ is Lipschitzian with the constant n, and therefore continuous on Y for each fixed $x \in X$.

It remains to prove that $f = \liminf_{h \to \infty} f_n$.

Let $(x,y) \in X \times Y$ be fixed and let $f(x,y) < -b$ for some positive $b \in R$. There exists an $r > 0$ such that $d(y,y_1) < r$ implies $f(x,y_1) < -b$ for $f(x,\cdot)$ being upper semicontinuous.

For each $n > b^{-1}$ we have

$$f(x,y_1) - n\, d(y,y_1) \leq f(x,y_1) < -b \quad \text{if} \quad d(y,y_1) < r$$

$$f(x,y_1) - n\, d(y,y_1) < f(x,y_1) - b \leq -b \quad \text{if} \quad d(y,y_1) \geq r .$$

Passing to the supremum of both sides of these inequalities we obtain $f_n(x,y) \leq -b$. Then taking into account that $f_n(x,y) \geq f(x,y)$ we deduce the equality $\inf_{n \in N} f_n = \liminf_{n \to \infty} f_n = f$. Let $\varepsilon > 0$ be an arbitrary, but fixed number. In compliance with theorem 2 and remark 1 for each f_n we shall find a compact subset K_n such that $m(X \setminus K_n) < 2^{-n}\varepsilon$ and the restriction $f_n \mid K_n \times Y$ is continuous on $K_n \times Y$. Put $K_\varepsilon = \bigcap_{n=1}^{\infty} K_n$ and observe that

$$m(X \setminus K_\varepsilon) = m\Big[\bigcup_{n=1}^{\infty}(X \setminus K_n)\Big] \leq \sum_{n=1}^{\infty} m(X \setminus K_n) = \sum_{n=1}^{\infty} 2^{-n}\varepsilon = \varepsilon .$$

Thus each f_n is continuous on $K_\varepsilon \times Y$. Consequently $f = \inf_{n \in N} f_n$ is upper semicontinuous on $K_\varepsilon \times Y$, as required.

Lemma 2. Let (X,m) be a Hausdorff topological space endowed with a finite, positive Radon measure m, Y a Polish space and F a multifunction from $X \times Y$ into the hyperspace of nonempty, closed subsets of some Polish space Z, such that:

(a) F is $M_m \otimes B(Y)$ -measurable on $X \times Y$

(b) $F(x,\cdot)$ is lower semicontinuous on Y for each fixed $x \in X$. Then for each $\varepsilon > 0$ there exists a compact subset $K_\varepsilon \subset X$ such that $m(X - K_\varepsilon) \leq \varepsilon$ and the restriction $F \mid K_\varepsilon \times Y$ is lower semicontinuous.

PROOF : Fix in Z a complete and bounded metric d, such that $d(Z \times Z) \leq 1$. Let $\{z_1, z_2, \dots\}$ be a countable, dense subset of Z. The lower semicontinuity of F is equivalent to the upper semicontinuity of single - valued functions $(x,y) \mapsto h_n(x,y) = \operatorname{dist}[z_n, F(x,y)]$. By virtue of (a), the function $(x,y) \mapsto \operatorname{dist}[z, F(x,y)]$ is $M_m \otimes B(Y)$-measurable for each fixed $z \in Z$. It follows from (b), is upper semicontinuous for each fixed $(x,z) \in X \times Z$. By lemma 1 for each $\varepsilon > 0$ we may select a compact subset $K_\varepsilon \subset X$ such that $m(X \setminus K)_\varepsilon \leq \varepsilon$ and that the restrictions of each h_n to $K \times Y$ are all upper semicontinuous. Thus, applying once again the lemma 0 we conclude that $F \mid K_\varepsilon \times Y$ is lower semicontinuous. Thus the proof of lemma 2 is completed.

3. Definitions and some examples of S-contractible spaces

Following L. Pasicki [15] (see also [18]) a set X is S-linear if $S: X\times[0;1]\times X \to X$ is a mapping such that $S(x,0,y)=y$ and $S(x,1,y)=x$ for all $x,y\in X$. For any subset $A\subset X$ we define

$$\mathrm{coS}\,A := \bigcap\{D\subset X : A\subset S(A\times[0;1]\times D)\subset D\},\ \text{where}$$

$$S(A\times[0;1]\times D) := \bigcup\{S(x,t,y): x\in A,\ y\in D,\ 0\leq t\leq 1\}.$$

If $A=\mathrm{coS}\,A$, then A is called S-convex.

A topological space X is S-contractible if X is S-linear, and for any $x\in X$, $\{S(x,t,\cdot)\ ; t\in[0;1]\}$ is a homotopy joing the identity with a constant map.

A topological space X is of C-type I if C is a subset of X and there exists an S such that X is S-contractible and for any $x\in C$ and any neighbourhood N of x there exists a neighbourhood U of x such that $\mathrm{coS}\,U\subset N$. If $C=X$ then we say it is of type I. Let (X,d) be a metric space. For the nonempty set $A\subset X$ and $r>0$ let us write $K(A,r) :=$

$$:=\{x\in X : \mathrm{dist}(x,A) := \inf_{a\in A} d(a,x)<r\}$$

A metric space (X,d) is uniformly of type 0 for balls if there exists an S such that for every $\varepsilon>0$ there exists a $\delta>0$ such that for every $A\in 2^X$ we have $\mathrm{coS}\,K(A,\delta)\subset K(\mathrm{coS}\,A,\varepsilon)$ and X is of type I for S.

Very recently Lech Pasicki was proved the following improvement of celebrated Michael's Selection Theorem [14]:

Theorem 3 [15] Let X be a paracompact topological space and (Y,d) a metric space uniformly of type 0 for balls. If the values of multifunction $G: X\to Y$ are complete, S-convex, and G is lower semicontinuous then G admits a continuous selector.

The following corollary is nicely harmonizing with a result of [18]:

Corollary: Let Y be a metric space uniformly of type 0 for balls and let A be a closed subset of a paracompact space X. Then every continuous function $f: A\to Y$ admits a continuous extension $E(f): X\to Y$ with $E(f)*X\subset \mathrm{Cl}[\mathrm{coS}(f*A)]$

P r o o f: Let $F : X \to Y$ be the S-convex carrier defined by $F(a) = \{f(a)\}$ for an $a \in A$; $F(x) = Y$ for $x \in X \setminus A$. From the continuity of f it follows that F is lower semicontinuous. Thus a selector for F from the theorem 3 is the desired extension $E(f)$ of f.

We give several examples. In Example 1 we argue that Bauer-Bear selection theorem is some special case of Pasicki theorem 3, collating adequate properties of part metric. Note, that this important and very natural example is not mentioned in Pasicki paper [15]. In Example 3 and 2 the S-contraction is defined on the space without any linear structure. Example 2 drawed from [15] shows that theorem 3 cannot be deduced from results of [1] nor [13]. Also Mägerl's unified selection theorem [12] is not more general, than theorem 3 since coS may fails to be an hull-operator in the sense of [12].

EXAMPLE 1. We consider a real linear space L and a convex set C in L which contains no whole line. We do not necessarily assume that L has a topology. Define $S: C \times [0;1] \times C \to C$ as follows:

$$S(x,t,y) := t \cdot x + (1-t) \cdot y .$$

Obviously C is S linear and coS = conv, the usual convex-hull operator.

The closed segment from x to y is denoted $[x,y] := \{S(x,t,y): 0 \leq t \leq 1\}$. If $x,y \in C$, we say that $[x,y]$ extends (in C) by $r > 0$ if $x + r(x-y) \in C$ and $y + r(y-x) \in C$. We write $x \sim y$ if $[x,y]$ extends by some $r > 0$. It is shown, [2], that $\sim$ defines an equivalence relation in C.

The equivalence classes of $\sim$, called the parts of C, are clearly also convex. There is a metric d on each part of C defined by

$$d(x,y) := \inf \{ \log(1 + r^{-1}) : [x,y] \text{ extends by } r \text{ in } C \}.$$

If $[x,y]$ extends by r in C, then $x + r_1(x-y)$ and $y + r_1(y-x)$ are in the part $[x] = [y]$ for all $r_1 < r$. It follows that one gets the same part metric on $[x]_\sim$ if one replaces C by $[x]_\sim$ in the definition of $d(x,y)$. If $(x,y) \notin \sim$, we write $d(x,y) = +\infty$. Then $d: C \times C \to R$ satisfies all axioms of a metric on C, except that it is not always finite. We could introduce $d_1 := \operatorname{arc\,tg} \circ d$ to obtain a metric on C defining the same topology as a generalized metric d. For each part $[x_1]$ of C and each $x \in [x_1]$ we have

$$[x_1] = \{y \in C : d(x,y) < \infty\}.$$

Therefore, the parts are open, and hence also closed.
Let $x, x_1, y, y_1 \in C$ and $t \in [0;1]$. If $[x, x_1]$ and $[y, y_1]$ extend by r, then $[S(x,t,y), S(x_1,t,y_1)]$ also extends by r. In fact, we have the identity

$$S(x,t,y) + r[S(x,t,y) - S(x_1,t,y_1)] = S(x + r(x-x_1),t,y + r(y-y_1)).$$

The term on the right is a convex combination of points which are in C by hypothesis. The extension beyond $S(x_1,t,y_1)$ follows by a symmetric argument. Notice that we do not assume that $x \sim y$ nor $x_1 \sim y_1$. As an immediate consequences we obtain the corollaries bellow :

(i) $d(S(x,t,y), S(x_1,t,y_1)) \leq \max[d(x,x_1), d(y,y_1)]$

(ii) If A is a convex subset of C (not necessarily in one part) and $d(x,A) < r$, $d(y,A) < r$, then

$$d(S(x,t,y), A) < r, \text{ even if } (x,y) \notin \sim .$$

(iii) The generalized d-ball $K(A,r)$ is convex for any convex subset A of C.

It is proved in [1, th. 3, p. 18] that for each part P of C, the mapping $S : P \times [0;1] \times P \to P$ is continuous, and that if C is an open convex set in a linear topological space, $0 \in C$ and C contains no line, then C has one part, moreover the part metric and the Minkowski norm $\|x\| = \operatorname{map}\{p(x), p(-x)\}$, where $p(x) = \inf\{r : x \in r \cdot C, r > 0\}$, define the same topology on C. It is easy to observe, that C in the general case, endowed with the topology generated by d is an S-contractible space uniformly of type 0 for balls.

EXAMPLE 2 ([15]) Let $X = J(\mathfrak{m})$ be a hedgehog with $\mathfrak{m}$ pricks, viz. the quotient space $X = M \times [0;1]/\sim$, where card $M = \mathfrak{m}$ and

$$(m_1,t_1) \sim (m_2,t_2) \text{ iff } t_1 = t_2 = 0 \text{ or } (m_1,t_1) = (m_2,t_2)$$

The formula

$$d([m_1, t_1]_\sim, [m_2,t_2]_\sim) = \begin{cases} |t_1 - t_2| & \text{if } m_1=m_2 \\ t_1 + t_2 & \text{if } m_1\neq m_2 \end{cases}$$

define a metric on X. Putting $S([m_1,t_1]_\sim, t, [m_2,t_2]_\sim) :=$

$$:= \begin{cases} [m_2, tt_1 + (1-t)t_2]_\sim \text{ if } 0 \leq t \leq 1 \text{ and } (m_1=m_2 \text{ or } t_1=0) \\ [m_2, -tt_1 + (1-t)t_2]_\sim \text{ if } 0 \leq t \leq \frac{t_2}{t_1+t_2}, m_1\neq m_2, t_1\neq 0 \\ [m_1, tt_1 - (1-t)t_2]_\sim \text{ if } \frac{t_2}{t_1+t_2} < t \leq 1, m_1\neq m_2, t_1\neq 0 \end{cases}$$

we obtain an S-contractible space uniformly of type 0 for balls, in which $K(A,r) = K(\text{co}S\ A, r)$.

EXAMPLE 3. Let X be a compact Riemannian manifold with Riemanian metric d. If $x,y \in X$ have a unique shortest geodesic joining them, this geodesic is called a segment. Let $L = \{(x,y) \in X\times X : x$ and y are joined by a segment$\}$ If $(x,y) \in L$, then the segment from x to y is given by a continuous $g_x^y : [0;1] \to X$. Define $S_1 : L \times [0;1] \times L \to$ by $S_1(x,t,y) := g_x^y(t)$. Every point p of X has a spherical neighbourhood $V(p)$ such that $V(p)$ is an S-contractible space uniformly of type 0 for balls, where $S = S_1 | V(p)\times[0,1]\times V(p)$ (cf. [13], p.569).

4. Caratheodory's selectors for F with values in Pasicki spaces

We are now in a position to state our main result:

Theorem 4. Let X be a Hausdorff topological space endowed with a finite, positive Radon measure m, Y a Polish space and F a multifunction from $X\times Y$ into the hyperspace of nonempty closed and S-convex subsets of some S-contractible Polish space Z, uniformly of type 0 for balls, such that:

(a) F is $M_m \otimes B(Y)$-measurable on $X \times Y$ and

(b) $F(x,\cdot)$ is lower semicontinuous on Y for each fixed $x \in X$.

Then there is a function $f : X \times Y \to Z$ such that:

(i) f_x is continuous on Y for each $x \in X$

(ii) f^y is m-measurable on X for each $y \in Y$

(iii) $f(x,y) \in F(x,y)$ for each $(x,y) \in X\times Y$.

Note, that the way treated by Ricceri [16] is not applicable since C(Y,Z) may fails to be an $\hat{S}$-contractible space for an $\hat{S}$ defined by a natural formula

$$\hat{S}(f_1,t,f_2)(z) := S(f_1(z),\ t,\ f_2(z)) \quad .$$

On the other hand, the way used by Fryszkowski [11] is also not adequate to our purpose, since the existence of continuous Castaing-Nowikow type representations for the (Z,S) - valued lower semicontinuous multifunctions is possible only under additional assumptions imposed on S (see [15])

P r o o f. Using (B) we shall find a compact $K_1 \subset X$ with $m(X \setminus K_1) \leq 2^{-1} m(X)$, $K_2 \subset X \setminus K_1$ with $m(X \setminus (K_1 \cup K_2)) \leq 2^{-1} m(X \setminus K_1)$ and so on. Proceeding inductively we obtain a partition $X = \bigcup_{n=1}^{\infty} K_n \cup N$ where each K_n is compact and $m(N) = 0$.

In accordance with lemma 1 the sets K_n may be choosen such that $F|K_n \times Y$ is lower semicontinuous for each n.

Let f_n be a continuous selector for $F|K_n \times Y$ existing in compliance with Pasicki theorem 3. Put

$$f(x,y) = \begin{cases} f_n(x,y) & \text{if } x \in K_n \\ s(x,y) & \text{if } x \in N \end{cases}$$

where $s(x,\cdot)$ is a continuous selector of $F(x,\cdot)$.

Observe that s is $M_m \otimes B(Y)$ measurable since N is a n-null subset. Consequently f is as required.

5. Caratheodory's selectors for F with values in Michael's convex structures

Let P_n denote the unit simplex in euclidean n-dimensional space R^n, i.e.

$$P_n := \left\{ t = (t_1,t_2,\dots,t_n) : 0 \leq t_i \leq 1,\ i=1,2,\dots,n \text{ and } \sum_{i=1}^{n} t_i = 1 \right\}$$

If E is any set, then E^n will denote the n-fold Cartesian product of E, and if $i \leq n$, then $\partial_i : E^n \to E^{n-1}$ is defined by $\partial_i(x_1,x_2,\dots,x_n) := (x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$.

A convex structure (cf. [13]) on a metric space E with metric d assigns to each positive integer n a subset M_n of E^n and a function $k_n : M_n \times P_n \to E$ such that :

(A) If $x \in M_1$, then $k_1(x,1) = x$

(B) If $x \in M_n$, $n \geq 2$ and $i \leq n$, then $\partial_i x \in M_{n-1}$ and, for any $t \in P_n$ with $t_i = 0$, $k_n(x,t) = k_{n-1}(\partial_i x, \partial_i t)$

(C) If $x \in M_n$ $(n \geq 2)$ with $x_i = x_{i+1}$ for some $i < n$, and if $t \in P_n$, then $k_n(x,t) = k_{n-1}(\partial_i x, t^*)$, where $t^* := (t_1, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_n)$

(D) If $x \in M_n$, then the map $t \mapsto k_n(x,t)$ from P_n to E is continuous

(E) For all $\varepsilon > 0$ there exists a neighbourhood V_ε of the diagonal Δ in E x E such that, for all n and all $(x^1, x^2) \in M_n \times M_n$, $(x_i^1, x_i^2) \in V_\varepsilon$ for $i = 1,2,\ldots,n$ implies

$$d(k_n(x^1, t), k_n(x^2, t)) < \varepsilon \text{ for all } t \in P_n .$$

Note that conditions (A) and (C) together imply that, if $x \in M_n$ with $x_1 = \ldots = x_n$, then $k_n(x,t) = x$ for all $t \in P_n$.
A subset A of a space E with convex structure is admissible if $A^n \subset M_n$ for all n. If A is admissible, then the convex hull of A, denoted by co A, is

$$\text{co } A := \bigcup_{n=1}^{\infty} \{k_n(x,t) : x \in A^n, t \in P_n\}$$

In [13, p.558], the following selection theorem is proved:
Theorem 5. Let Y be a complete metric space with a convex structure, let X be paracompact and F: X → Y lower semicontinuous multifunction. Suppose that F(x) is nonempty, admissible subset of Y for each $x \in X$. Then there exists a continuous f:X → Y such that $f(x) \in \mathrm{Cl}\,[\mathrm{co}\, F(x)]$ for all $x \in X$.
Note, that the relationship between theorems 3 and 5 are nuclear. Let E be a metric space with convex structure. The a subset $A \subset E$ is said to be convex, if it is admissible and co $A \subset A$. Repeating the proof of theorem 4 with theorem 5 invoked instead of th.3 we obtain:
THEOREM 6. Conserving all assumptions and notations of theorem 4 assume that Z is a Polish space with Michael's convex structure, and that F(x,y) is nonempty, closed, convex subset of Z for each $(x,y) \in X \times Y$. Then there is a Caratheodory's selector f : X x Y → Z for F, i.e. a funct-

ion satisfying conditions (i) - (iii) from theorem 4.

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ABSTRACT

We obtain two theorems on existence of separately continuous and separately measurable selectors for certain multifunctions defined on product spaces and taking values in so-called S-contractible complete metric spaces of type 0 uniformly for balls [th.4] or in spaces endowed with so-called Michael's abstract convex structures [th. 6]. Moreover we give an example showing that Heinz-Bauer selection theorem is some special case of known Pasicki selection theorem.

O SELEKTORACH CARATHEODORYEGO DLA MULTIFUNKCJI
O WARTOŚCIACH W PRZESTRZENIACH S-ŚCIAGALNYCH

Streszczenie

Praca poświęcona jest dowodowi dwu twierdzeń dotyczących istnienia selektorów mierzalnych ze względu na jedną zmienną i ciągłych ze względu na drugą zmienną z osobna dla multifunkcji przybierających wartości wypukłe w przestrzeniach S-ściągalnych typu O wprowadzonych przez Pasickiego oraz w abstrakcyjnych strukturach wypukłych wprowadzonych przez Michaela. Ponadto dyskutujemy związki twierdzenia Bauera z ogólniejszym twierdzeniem Pasickiego.