ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ w BYDGOSZCZY
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ON CEDERS CONTINUITY PROPERTY AND BAIRE I SELECTORS

It is well-known that if b and t are real-valued functions defined on a perfectly normal topological space, with $b \leq t$, then there exists a continuous function f such that $b \leq f \leq t$ provided b is upper semi-continuous and t is lower semicontinuous (see [1],[5]-[7],[11],[13]). More generally, a lower semi-continuous set-valued function F from any perfectly normal space into the hyperspace of nonempty convex subsets of the real line R admits a continuous selector f (i.e. $f(x) \in F(x)$ for all x), [10].

The purpose of this paper is to characterize those set-valued mappings from a given perfectly normal space into the family of non-empty intervals of R which admit continuous selectors. As a consequence we obtain characterizations for the insertion of continuous function between two comparable functions, in case $b \leq t$. Our theorem 1 can be vieved as an improvement of Ceders characterization([3], th. 1 ;[2], th. 1).

Using this improvement we are able to generalize the main result of the paper [2] onto the case of multifunctions defined on an euclidean space R^N. This solves some problem posed by J. Ceder in [4]. Let us recall that a real-valued function f on X is said to be lower semicontinuous (briefly lsc) (resp. upper semicontinuous = usc) provided for all $x \in X$

$$\liminf_{z \to x} f(z) \geq f(x)$$

(resp. $\limsup_{z \to x} f(z) \leq f(x)$).

Some useful facts about semi-continuous functions are (cf.[ ]):

(1) f is lsc (resp usc) if and only if $\{x : f(x) > a\}$ (resp. $\{x : f(x) < a\}$ is open for each $a \in R$;

(2) a lsc (resp. usc) function achieves its minimum (resp. maximum) on each compact set ;

(3) the minimum (resp. maximum) of two lsc (resp. usc) functions is again lsc (resp. usc)

(4) the set of continuity points of a semicontinuous function is residual in X .

A set-valued mapping F from any topological space X into the family of nonvoid subsets of a topological space Y is said to be lower semicontinuous if $F^{-}(V) := \{x \in X: F(x) \cap V \neq \emptyset\}$ is open in X for every open V in Y . It is easily seen that if $f \leq g$ on X and f is usc, and g is lsc, then F is lsc, where $F(x) := [f(x), g(x)]$ cf. [10], Ex. 1.2., p.362). We will always identify a function with its graph. By $f \mid A$ we mean the restriction of f to A . By C(f) is meant the set of continuity points of f . We denote for any f and x

$$f_*(x) = \liminf_{z \to x} f(z) \quad \text{and} \quad f^*(x) = \limsup_{z \to x} f(z)$$

THEOREM 1. Suppose $F : X \to R$ is a set-valued mapping from a perfectly normal space X with non-empty convex subsets of the real line R as values. Then, there exists a continuous selector for F if and only if for all $x \in X$

(i) $b^*(x) := \limsup_{z \to x} b(z) \leq \liminf_{z \to x} t(z) := t_*(x)$;

(ii) $F(x) \cap [b^*(x), t_*(x)] \neq \emptyset$

where b(x) and t(x) are the inf and sup of F(x) respectively.

Proof. Suppose f is a continuous selector for F. Then clearly $b^*(x) \leq f(x) \leq t_*(x)$ from which both (i) and (ii) follow. Now suppose (i) and (ii) hold. Define $G(x) := F(x) \cap [b^*(x), t_*(x)] =: [k(x), l(x)]$, and observe that G has nonempty convex values. It is easy to verify that sup $G(x) = l(x)$ is lsc. In fact, for $x \in X$ either $l(x) = t_*(x)$ or $l(x) = t(x)$ and $t_*(x) \geq t(x)$. In the first case l is semicontinuous at x by virtue of [14] , lemme V.1.4. , p.136. In either case l is lsc simply by definition. In a similar manner we can establish the upper semicontinuity of k = inf G. Therefore, by Ex 1.2., p.362 of [10], G is lsc as a convex-valued

multifunction .
By [10], th. 3.1''', p. 308 on can select a continuous selector f for G . Observe that $f(x)\in G(x)\subset F(x)$. This completes the proof of theorem 1 .

Corollary 1. Suppose $f\leq g$ on a perfectly normal space X. Then there exists a continuous function h such that $f\leq h\leq g$ if and only if for all $x\in X$: (i) $f^*(x)\leq g_*(x)$
(ii) $[f(x), g(x)]\cap[f^*(x), g_*(x)]\neq\emptyset$

Corollary 2. Suppose $f<g$ on a perfectly normal space X . Then there exists a continuous function h such that $f<h<g$ if and only if for all $x\in X$
(i) $f^*(x)\leq g_*(x)$
(ii) $(f(x), g(x))\cap[f^*(x), g(x)]\neq\emptyset$

Since it is easy to verify that a lsc F satisfies conditions (i) and (ii) of the theorem 1 we also have Michael's result as a corollary. For further informations about insertion of a continuous function see [1],[5-7],[11],[13]. It is unknown whether or not can one generalize the range of F to some nice family of sets (e.g. the open disks in R^2) and obtain some reasonable characterization for the admission of a continuous selector .

There are already some theorems in which the condition to impose upon a multifunction for the admission of a nice selector is that the multifunction restricted to each of a family of small sets has a nice selector. A result of this kind is the following :

THEOREM 2 (Lindenstrauss [9] , cf. also [8])
Let M be a metric space and let B be a Banach space. Let $F: M\to B$ be a multifunction such that F(m) is closed, convex and separable subset of B for every $m\in M$. Assume that for every countable compact subset K of M the restriction $F|K$ of F to K admits a continuous selector on K . Then F admits a continuous selector .

Another result of this kind is the following

THEOREM 3 (Ceder [2] , cf. [8],[4])
Let $F: R\to R$ be a multifunction such that F(x) is closed

and convex for every $x \in R$. Then F has a Baire 1 selector if and only if $F|P$ has a Baire 1 selector for each perfect, nowhere dense subset P of R.

Note that paper [2] errorously claims, that in theorem 3, for insure the existence of Baire 1 selector it suffices to assume that $F|P$ has a Baire 1 selector for each perfect, nowhere dense subset of measure zero only.

Paper [4] posses the problem of generalizing the domain in this theorem. In order to solving this problem we need the following generalization of famous Baire theorem :

THEOREM 4 . A function $f : R^N \to R$ from the N-dimensional euclidean space into the real line is Baire 1 if and only if the restriction $f|P$ has a point of continuity for each perfect nonwheredense subset P of R^N.

Proof: Assume that $f|H$ is totally discontinuous for some perfect subset $H \subset R^N$. Thus $H = \bigcup_{n=1}^{\infty} H_n$, where $H_n := \{x: \text{osc } f \geq n^{-1}\}$. It is easily verified that each H_n is closed. By the Baire Category Theorem there exists k such that H_k contains an open ball, say J, relative to H. We will contruct a countable subset D of J such that cl D is perfect and nowhere dense in J and such that for all $x \in \text{cl } D$ $\text{osc}(f|\text{cl } D) \geq k^{-1}$. Pick $d \in J$ and sequences d_n^1 and d_n^2, $n=1,2,\ldots$ in $J - \{d\}$ approaching d such that $\lim_{n\to\infty} f(d_n^1) = \limsup_{v \to d} f(v)$ and $\lim_{n\to\infty} f(d_n^2) = \liminf_{z\to d} f(z)$ and $d_n^1 \neq d_m^2$ for all n and $m \neq n$. Let D_1 consist of d and the terms of those sequences. For $z \in D_1 - \{d\}$ define $r(z) = 3^{-1}\,\text{dist}(z, D_1 - \{z\})$. Pick sequences z_n^1, z_n^2, $n = 1,2,\ldots$ in $K(z, r(z)) := \{z_1 \in J : \|z_1 - z\| < r(z)\}$, approaching z such that $z_n^1 \neq z_m^2$ for all $n \neq m$ $\lim_{n\to\infty} f(z_n^1) = \limsup_{v\to z} f(v)$ $\lim_{n\to\infty} f(z_n^2) = \liminf_{v\to z} f(v)$. Define :

$$D_2 := D_1 \cup \bigcup_{k=1}^{2} \bigcup_{n=1}^{\infty} \{z_n^k : z \in D_1 - \{d\}\}.$$

Now, continuing by induction in the obvious way we obtain a sequence of sets D_n. Putting $D := \bigcup_{n=1}^{\infty} D_n$ we have cl D is perfect and nowhere dense in J. In fact, each point of $\mathrm{cl}\, D$ is an accumulation point and in each relatively open subset $J_1 \subset J$ there is an relatively open ball $J_2 \subset J_1$ such that $J_2 \cap \mathrm{cl}\, D = \emptyset$. Moreover, it is easy to verify that for each $x \in \mathrm{cl}\, D$, osc $(f \mid \mathrm{cl}\, D) \geqslant k^{-1}$. The rest of the proof is obvious, by usual Baire theorem.

It would be interesting to know wheter this theorem remains true if P is assumed to be e.g. sigma-porous perfect set or at least nowhere dense null (i.e. $m_N(P) = 0$) perfect set.

THEOREM 5 . Let $F : R^N \to R$ be a multifunction with closed, convex values. Then F admits a Baire 1 selector if and only if for each perfect nowhere dense subset P of R^N , the restriction $F|P$ admits a Baire 1 selector.

Proof: First note, that the implication "only if" is obvious when F has a Baire 1 selector. So let us assume that $F|P$ has a Baire 1 selector for each perfect, nowhere dense subset P of R^N . Put $t(x) = \sup F(x)$ and $b(x) = \inf F(x)$. Define $t_*(x) := \liminf_{z \to x} t(z)$ and $b^*(x) := \limsup_{z \to x} b(z)$.

STEP 1: Construction of P_0 and f_0 :

Observe that $\{x \in R^N : b^*(x) < t_*(x)\} = (b^* - t_*)^{-1}((-\infty; 0))$ is open in R^N . It follows that there exists a nonempty open subset G with $R^N - G$ perfect such that either

(i) $\mathrm{cl} \subset G\, \{x : b^*(x) < t_*(x)\}$ or

(ii) $\mathrm{cl} \subset G\, \{x : b^*(x) \geqslant t_*(x)\}$. Put $P_0 := R^N - G$.

In case (i) let $D = \{x \in G : F(x) \cap [b^*(x) : t_*(x)] = \emptyset\}$.

Then D is finite. Indeed, if D were infinite, there would exist a sequence x_i, $i=1,2,\dots$ in G, and a point x $\mathrm{cl}\, G$ and $a, b \in R \cup \{-\infty, +\infty\}$ for which $b(x_i) \to a$, $t(x_i) \to b$, $x_i \to x$ and $F(x_i) \cap [b^*(x_i), t_*(x_i)]$ is empty. Without loss of generality we may assume that $t_*(x_i) < b\,(x_i)$. Then we must have $a \leq b^*(x) < t_*(x) \leq b$. On the other hand

$$t_*(x) \leq \liminf_{i \to \infty} t_*(x_i) \leq \limsup_{i \to \infty} t_*(x_i) \leq \lim_{i \to \infty} b(x_i) = a$$

This leads to a contradiction.

Our multifunction F has the continuity property (i.e. fulfills (i) and (ii) of th. 1) at each point of $G - D$. Therefore, by theorem 1 there is a continuous selector h for $F|G - D$. Now define

$$f_o(x) := \begin{cases} h(x) & \text{if } x \in G - D \\ \text{midpoint } F(x) & \text{if } x \in D \end{cases}$$

Clearly f_o is a Baire 1 selector for $F|(R^N - P_o)$.

In case (ii) let $E := \{x \in \text{cl } G : t_*(x) = b^*(x) \text{ and } b^*(x) \notin F(x)\}$. It is easy to verify that this set E is countable. In fact, let $x \in E$ so that $b(x) > \limsup_{t \to x} b(t)$. There is a basic open set $V(x) \subset \text{cl } G$ containing x and a basic open set $U(x) \subset R$ containing $b(x)$ such that $b(t) \notin U(x)$ for $t \in V(x) - \{x\}$. Observe that $(U(x_1), V(x_1)) \neq (U(x_2), V(x_2))$ whenever $x_1 \neq x_2$. Since the set of all pairs of basic open sets (in separable cl G and R) is countable, hence the set E is countable as well. Let $H = \{x : b^*(x) \neq t_*(x)\} = \{x : t_*(x) < b^*(x)\}$. Then H is a first category F_σ subset of G. In fact, let $H_n = \{x : b^*(x) - t_*(x) \geq n^{-1}\}$.

Since $u = b^* - t_*$ is upper semicontinuous function, it is easily seen that each $H_n = u^{-1}([n^{-1}; \infty))$ is closed and $H = \bigcup_{n=1}^{\infty} H_n$. If some H_n is dense somewhere, say in $U \subset H_k = \text{cl } H_k$, then osc $f(x) \geq k^{-1}$ on U for each selector f of our multifunction F. In fact, we have osc $f = f^* - f_* \geq b^* - t_*$ for $b(x) \leq f(x) \leq t(x)$. Thus any selector f cannot be of the first Baire class on U. By virtue of th. 4 there is a nowhere dense perfect subset $D \subset U$ such that $f|D$ is totally discontinuous on D. But this is in marked contrast with assumption, that $F|D$ must have a Baire 1 selector. Hence H is an F_σ of the first category relative to G let $A_1 = H_1$, $A_n := H_n - H_{n-1}$ for $n = 2, 3, \ldots$, Each A_n is ambiguous and we have $A_n \cap A_m = \emptyset$ when $n \neq m$. Moreover $H = \bigcup_{n=1}^{\infty} H_n = \bigcup_{n=1}^{\infty} A_n$.

By the condition we may choose a Baire 1 selector f_n for $F|H_n$. Now define for $x \in G$:

$$f_o(x) := \begin{cases} f_n(x) & \text{if } x \in A_n \\ b^*(x) & \text{if } x \in G - E - H \\ 2^{-1} \cdot [t(x) + b(x)] & \text{if } x \in E. \end{cases}$$

Observe that $G - E - H \subset C(f_o)$, the set of continuity points of f_o. In fact for $x \in G - E - H$, $f_o(x) = b^*(x) = t_*(x)$, so that $f_o : R^N - P_o \to R$ is simultaneously lsc and usc at x. Observe that $E \cap C(f_o) = \emptyset$ for $\lim_{n\to\infty} \sup b(x_n) < b^*(x) = t_*(x) \leq \leq \lim_{n\to\infty} \inf t(x_n)$ whenever x_n tends to x in E. Also if $x \in H$, then there exist cluster values 1 and m of f_o such that $m \leq t_*(x) < b^*(x) \leq 1$.
Hence $H \cap C(f_o) = \emptyset$. Therefore $C(f_o) = G - E - H$ and $(f_o | G - E - H)^{-1}(U) = \{x \in R^N : x \in G - H$ and $f_o(x) \in U\}$ is open in R^N (and hence in $R^N - P_o$) for each open set $U \subset R$.
We have $f_o^{-1}(U) = (G - E - H) \cap (b^*)^{-1}(U) \cup \bigcup_{n=1}^{\infty} [f_n^{-1}(U) \cap A_n] \cup$
$\cup \{x : 2^{-1} \cdot [t(x) + b(x)] \in U\} \in F_\sigma(R^N - P_o)$.
and thus f_o is a Baire 1 selector for $F|G = F|(R^N - P_o)$.
Denote by Ω the first uncountable ordinal number and let $\beta < \Omega$
Using transfinite induction, suppose we have constructed for each $\beta < \Omega$ sets P_α and functions f_α such that

(u1) P_α is a perfect set

(u2) f_α has domain $R^N - P_\alpha$

(u3) $P_\delta \subset P_\alpha$ whenever $\alpha < \delta$

(u4) $f_\alpha \subset f_\delta$ whenever $\alpha < \delta$

(u5) f_α is a Baire 1 selector for $F|R^N - P_\alpha$

(u6) $P_\alpha \neq \emptyset$ and $\alpha < \delta$ imply $P_\delta \neq P_\alpha$.

STEP 2 : Construction of f_β and P_β in general :
In case when $\beta = \xi + 1$ for $\xi < \Omega$ we construct a function h and a perfect set P_β in exactly the same way we construct f_o and P_o were P_ξ plays the role of the domain R^N in that construction. Note that when $P_\xi \neq \emptyset$, then $P_{\xi+1}$ is a proper subset of P_ξ in that construction. Define $f_{\xi+1}$ as a function $f_\xi \cup h$. Since f is assumed to be a Baire 1 function on the open set $R^N - P_\xi$ and h is a Baire 1 function on the F_σ set $P_\xi - P_{\xi+1}$, it follows that f_β is a Baire 1 function on $R^N - P_{\xi+1}$. The remaining conditions of the inductive hypothesis are clear.
In case when β is a limit ordinal, observe, that the set $\bigcap_{\alpha<\beta} P_\alpha$ is closed and therefore, by famous Cantor-Bendixon theo-

rem, is the union of a perfect set P and a countable set C with $P \cap C = \emptyset$. Put $P_\beta = P = P^{\odot}$ and define f_β on $R^N - P_\beta$ as follows :

$$f_\beta(x) := \begin{cases} \text{midpoint } F(x) \text{ if } x \in C \\ f_\alpha(x) \text{ if } x \in R^N - P_\alpha \text{ for some } \alpha \end{cases}$$

Then $f_\beta = (\bigcup_{\alpha<\beta} f_\alpha) \cup (f_\beta | C)$ and the domain of f_β is the open set $R^N - P_\beta$. To show that f_β is Baire 1 we need only show that $f_\beta | Q$ has a point of continuity for each perfect set $Q \subset R^N - P_\beta$. Since card $Q = \mathfrak{c}$ it must intersect some Dom f_α for $\alpha < \beta$. Hence a portion of Q is contained in the open set $R^N - P_\beta$ upon which f_β is Baire 1 .
Therefore $f_\beta | Q$ has a point of continuity in $R^N - P_\beta$.
The rest of the inductive hypotheses are easily verified.
Therefore, by transfinite induction there exists a descending chain of perfect sets $\{P_\alpha , \alpha < \Omega\}$ and an ascending chain of functions $\{f_\alpha : R^N - P_\alpha \to R ; \alpha < \Omega\}$ such that for each α

(a) f_α has domain $R^N - P_\alpha$ and is a Baire 1 selector for $F | R^N - P_\alpha$, and

(b) $P_\delta \neq P_\alpha$ whenever $P_\alpha \neq \emptyset$ and $\delta > \alpha$.

Since $\{P_\alpha ; \alpha < \Omega\}$ is a decreasing chain of closed sets it is eventually constant, that is, there is a γ such that $P_\xi = P_\gamma$ whenever $\gamma < \xi$. By (b) we must have $P_\gamma = \emptyset$. Therefore, by (a) , f_γ is the desired Baire 1 selector for F on R^N .
This finishes the proof of theorem 5 .
As a corollary we obtain :
THEOREM 6. Let $F: R^N \to R$ be a multifunction from an euclidean space R^N into the hyperspace of non-void, closed convex subsets of the real line. Then F admits a Baire 1 selector if and only if for each nowheredense perfect subset P of R^N ,the restriction $F|P$ has the continuity property at some point of P.
Both theorems 5 and 6 apply only to those F for which each $F(x)$ is simultaneously closed and convex :
THEOREM 7 (cf. [12]) There exists a multifunction F $R^N \to R$ with non-void, convex values, admitting on Baire 1 selectors but with the property, that the restriction of this multifunct-

ion to an arbitrary perfect subset P R^N has the continuity property at some point of P.

Proof: Let Z be a totally imperfect Berstein set (see [D], th. 1) in R^N which intersects P and $R^N - P$ for each perfect subset $P \subset R^N$. Put :

$$F(x) := \begin{cases} (0, 1] & \text{if } x \in Z \\ (-1, 0] & \text{if } x \in R^N - Z \end{cases}$$

and observe that $t := \sup F = I_Z$, $b := \inf F = - I_{R^N - Z}$. Thus we have $t_* = b^* = 0$ identically on R^N. Fix some perfect set $P \subset R^N$ and note that the intersection $(R^N - Z) \cap P$ is nonempty. Let x_0 be some element of this intersection. Since 0 belongs to $F(x_0)$, it follows that $F(x_0) \cap [b^*(x_0), t_*(x_0)] = \{0\} \neq \emptyset$ and thus F has the continuity property at selected point $x_0 \in P - Z$. Observe that if f is any selector for F, then the inverse image $f^{-1}((0,2)) = Z$ is not Borel set despite (0,2) is open . This completes the proof,

Our theorems 4 nor 5 does not carry over to the case of higher Baire classes. In fact, we have :

THEOREM 8. Assume continuum hypothesis. There is a function $f : R^N \rightarrow R$ such that for each perfect, nowheredense subset D of R^N the restriction $f | D$ is of the second Baire class, while f as not even Borel-measurable.

Proof: This follows for instance from [D], th.4.

After this paper has been completed, the author learned about the paper by Vetro Pasquale [V] where the theorem very similar to our theorem 1 is also proved. To author wishes to express his thanks to Prof. J.S. Lipiński for his critical remarks.

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O CEDERA WŁASNOŚCI CIĄGŁOŚCI

Streszczenie

H. Ceder w [2] podał charakteryzację tych multifunkcji $F : R \rightarrow R$ o wypukłych wartościach, które posiadają ciągły selektor i wykorzystał ten wynik do dowodu istnienia selektora pierwszej klasy Bairea dla multifunkcji $F : R \dashrightarrow R$ o domkniętych wypukłych wartościach, o których wiadomo, że po obcięciu do każdego nigdziegęstego zbioru doskonałego posiadają taki selektor. W niniejszym artykule uogólnia się te wyniki na przypadek, gdy dziedziną jest dowolna skończeniewymiarowa przestrzeń euklidesowa, rozwiązując w ten sposób pewien problem Cedera. Ostatnie twierdzenie, mówiące o tym, że w przypadku wyższych klas Bairea sytuacja jest całkowicie odmienna podano bez dowodu, gdyż wynika ono z pracy zamieszczonej w tymże zeszycie, opracowanej przez Koło Naukowe studentów. Dla kompletności przytoczono informację o istnieniu ciągłych selektorów dla multifunkcji o których wiadomo że posiadają ciągły selektor po obcięciu do każdego przeliczalnego podzbioru zwartego przestrzeni metrycznej .