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MAREK BALCERZAK
Uniwersytet Łódzki

A CLASSIFICATION OF σ-IDEALS ON THE REAL LINE

Throughout the paper we shall consider subsets of the real line $\mathbb{R}$ equipped with the natural topology. By ω (resp. ω_1) we mean the first infinite (resp. uncountable) ordinal number. Let $\mathcal{B}$ denote the family of all Borel sets. We shall also consider families F_α, G_α, $\alpha < \omega_1$, defined as in [2], pp.251--252.

A family $\mathcal{J}$ of sets will be called a σ-ideal if and only if it fulfils the conditions:

(i) if $A \in \mathcal{J}$ and $B \subseteq A$, then $B \in \mathcal{J}$;

(ii) if $A_n \in \mathcal{J}$ for all $n < \omega$, then $\bigcup_{n<\omega} A_n \in \mathcal{J}$;

(iii) if $A \in \mathcal{J}$, then the interior of A is empty;

(iv) if $x \in \mathbb{R}$, then $\{x\} \in \mathcal{J}$.

A family $\mathcal{J}$ will be called movable if and only if it fulfils the condition

(v) if $A \in \mathcal{J}$ and $x \in \mathbb{R}$, then $A + x \in \mathcal{J}$,

where $A + x = \{y \in \mathbb{R} : y = a + x$ for some $a \in A\}$.

<u>Remark 1.</u> If $\mathcal{J}$ is movable, conditions (i), (ii) hold and $\mathbb{R} \notin \mathcal{J}$, then conditions (iii), (iv) hold, as well.

Let $\mathcal{J}$ be a σ-ideal and let $\mathcal{T}$ be any of the families $\mathcal{B}$, F_α, G_α, $\alpha < \omega_1$. Define

$$I(\mathcal{J}, \mathcal{T}) = \{A : A \subseteq B \text{ for some } B \in \mathcal{J} \cap \mathcal{T}\}.$$

$\mathcal{J}$ will be called a Borel (resp. non-Borel) σ-ideal if and only if $I(\mathcal{J}, \mathcal{B}) = \mathcal{J}$ (resp. $I(\mathcal{J}, \mathcal{B}) \neq \mathcal{J}$).

We define $RF(\mathcal{J})$ (resp. $RG(\mathcal{J})$) as the first ordinal number $\gamma \leq \omega_1$ such that $I(\mathcal{J}, \mathcal{B}) = I(\mathcal{J}, F_\gamma)$ (resp. $I(\mathcal{J}, \mathcal{B}) = I(\mathcal{J}, G_\gamma)$). Here $F_{\omega_1} = G_{\omega_1} = \mathcal{B}$. We shall say that the σ-ideal $\mathcal{J}$ is of type $(\alpha; \beta)$ if and only if $\alpha = RF(\mathcal{J})$ and $\beta = RG(\mathcal{J})$.

<u>Lemma 1.</u> If $\mathcal{J}$ is a σ-ideal of type $(\alpha; \beta)$, then $\alpha = \beta$ or $\beta = \alpha + 1$ or $\alpha = \beta + 1$.

Proof. Suppose that $\alpha \neq \beta$, and let for example $\alpha < \beta$. We have

$$I(\mathcal{J},\mathcal{B}) = I(\mathcal{J},F_{\alpha}) \leq I(\mathcal{J},G_{\alpha+1}) \subseteq I(\mathcal{J},\mathcal{B}) .$$

Thus $I(\mathcal{J},G_{\alpha+1}) = I(\mathcal{J},\mathcal{B})$ and by the definition of β , we have $\beta \leq \alpha + 1$, which together with $\alpha < \beta$ gives $\beta = \alpha + 1$. In the case $\beta < \alpha$ the proof is analogous.

<u>Lemma 2</u>. If $\mathcal{J}$ is a σ-ideal of type $(\alpha ; \beta)$, then $\alpha > 1$ or $\beta > 1$.

<u>Proof</u>. Suppose that $\alpha \leq 1$ and $\beta \leq 1$. Since $\alpha \leq 1$, we have $I(\mathcal{J},\mathcal{B}) = I(\mathcal{J}, F_1)$. Hence, from (iii) and the definition of $I(\mathcal{J},F_1)$ it easily follows that all sets from $\mathcal{J} \cap \mathcal{B}$ are of the first category. Since $\beta \leq 1$, we have $I(\mathcal{J},\mathcal{B}) = I(\mathcal{J},G_1)$. In virtue of (iv), (ii), the set W of all rational numbers belongs to $I(\mathcal{J},\mathcal{B})$. So, by the definition of $I(\mathcal{J},G_1)$, there exists a set $B \in \mathcal{J} \cap G_1$ such that $W \subseteq B$. The set B belongs to $\mathcal{J} \cap \mathcal{B}$, so it is of the first category. But the Baire Category Theorem easily implies that the set of type G_δ and of the first category is nowhere dense. This gives a contradiction since B cannot simultaneously be nowhere dense and contain W .

From Lemmas 1,2 we immediately obtain the following

<u>Theorem 1</u>. If $\mathcal{J}$ is a σ-ideal of type $(\alpha ; \beta)$, then

$$(*) \quad 2 \leq \alpha = \beta \leq \omega_1 \quad \text{or} \quad 2 \leq \alpha + 1 = \beta < \omega_1 \quad \text{or}$$
$$2 \leq \beta + 1 = \alpha < \omega_1 .$$

Conversely, we shall prove (see Theorem 2 below) that if a pair α , β fulfils condition $(*)$, then there exists a σ-ideal $\mathcal{J}$ of type $(\alpha ; \beta)$. Thus, condition $(*)$ characterizes the type of σ-ideals.

Denote by $\mathcal{K}$ and $\mathcal{L}$ respectively, the σ-ideal of all sets of the first category and the σ-ideal of all sets of the Lebesgue measure zero. It is easily checked that $\mathcal{K}$ and $\mathcal{L}$ are Borel σ-ideals of types $(1;2)$ and $(2;1)$, respectively.

Let $\mathcal{L}_1 = I(\mathcal{L},F_1)$. Notice that it is a Borel σ-ideal of type $(1;2)$. We obviously have $\mathcal{L}_1 \subseteq \mathcal{K} \cap \mathcal{L}$. Let A be a closed nowhere dense set of positive measure and let $B \subseteq A$ be a set of type G_σ such that B belongs to $\mathcal{L}$ and contains a countable dense subset of A. Then we easily observe that $B \in (\mathcal{H} \cap \mathcal{L}) \setminus \mathcal{L}_1$.

Proposition 1. $\mathcal{H} \cap \mathcal{L}$ is a Borel σ-ideal of type 2;2 .

Proof. Let $\mathcal{H} \cap \mathcal{L}$ be of type $(\alpha;\beta)$. Since clearly

$$RF(\mathcal{H} \cap \mathcal{L}) \le \max(RF(\mathcal{H}), RF(\mathcal{L})),$$
$$RG(\mathcal{H} \cap \mathcal{L}) \le \max(RG(\mathcal{H}), RG(\mathcal{L})),$$

therefore $\alpha \le 2$, $\beta \le 2$. Let $\mathcal{N}$ denote the family of all nowhere dense sets. We have

$$I(\mathcal{H} \cap \mathcal{L}, F_1) = I(\mathcal{H}, F_1) \cap I(\mathcal{L}, F_1) = \mathcal{H} \cap \mathcal{L}_1 = \bar{\mathcal{L}}_1 \ne \mathcal{H} \cap \mathcal{L},$$
$$I(\mathcal{H} \cap \mathcal{L}, G_1) = I(\mathcal{H}, G_1) \cap I(\mathcal{L}, G_1) = \mathcal{N} \cap \mathcal{L} \ne \mathcal{H} \cap \mathcal{L},$$

thus $\alpha \ge 2$, $\beta \ge 2$, which ends the proof.

Now, we are going to give a few examples of non-Borel σ-ideals.

In the sequel, we shall always assume that a perfect set is nonempty.

Recall that a totally imperfect set means a set which does not contain any perfect set (comp. [2], p.421).

If $\mathcal{C}^1$, $\mathcal{C}^2$ are families of sets, then denote

$$\mathcal{C}^1 \oplus \mathcal{C}^2 = \{A_1 \cup A_2 : A_1 \in \mathcal{C}^1, A_2 \in \mathcal{C}^2\}.$$

Proposition 2. Let $\mathcal{J} = \mathcal{H}$ or $\mathcal{J} = \mathcal{L}$. Let $\mathfrak{I}^1$ be a family of sets such that

(1) $\mathfrak{I}^1$ fulfils conditions (i), (ii) ;

(2) $\mathfrak{I}^1$ consists of totally imperfect sets;

(3) there is a set $A \in \mathfrak{I}^1 \setminus \mathcal{J}$.

Let $\mathfrak{I}^2$ be a σ-ideal included in $\mathcal{J}$ and let $\mathfrak{I} = \mathfrak{I}^1 \oplus \mathfrak{I}^2$.

Then we have:

(a) $\mathfrak{I}$ is a non-Borel σ-ideal;

(b) if $\mathfrak{I}^1$, $\mathfrak{I}^2$ are movable, so is $\mathfrak{I}$;

(c) if $\mathfrak{I}^2$ is a Borel σ-ideal, then $I(\mathfrak{I}, \mathcal{B}) = \mathfrak{I}^2$ and $\mathfrak{I}, \mathfrak{I}^2$ are of the same type.

Proof. (a) Conditions (i), (ii), (iv) of the definition of a σ-ideal are easy to verify . It remains to prove (iii). Suppose that there is an open interval $U \in \mathfrak{J}$. Then there exist sets $A_1 \in \mathfrak{J}^1$, $A_2 \in \mathfrak{J}^2$ such that $U \subseteq A_1 \cup A_2$. Let $B \in \mathfrak{Y}$ be a Borel set such that $A_2 \subseteq B$. Then $U \setminus B$ is Borel and uncountable, so, in virtue of the Alexandroff--Hausdorff theorem (see [2], p.355), it contains a perfect set C . Then $C \subseteq A_1$ which contradicts (2) . Thus (iii) hold and $\mathfrak{J}$ is a σ-ideal. To prove that $\mathfrak{J}$ is non-Borel, observe that $A \in \mathfrak{J}$ and $A \notin I(\mathfrak{J}, \mathfrak{B})$. The former relation is obvious. To prove the latter, suppose that $A \in I(\mathfrak{J}, \mathfrak{B})$. Then there is a set $B \in \mathfrak{B} \cap \mathfrak{J}$ such that $A \subseteq B$. Let $B = B_1 \cup B_2$ where $B_1 \in \mathfrak{J}^1$, $B_2 \in \mathfrak{J}^2$. We may assume that B_1, B_2 are disjoint. The set $B_1 = B \setminus B_2$ has the Baire property or is Lebesgue measurable since $B \in \mathfrak{B}$ and $B_2 \in \mathfrak{J}^2 \subseteq \mathfrak{Y}$. Moreover, $B_1 \notin \mathfrak{Y}$ since, in the contrary case, we would have $A \in \mathfrak{Y}$, which contradicts (3) . Thus B_1 contains a Borel uncountable set. So it has a perfect subset and this contradicts (2) . Therefore $A \notin I(\mathfrak{J}, \mathfrak{B})$.

Statement (b) is self-evident.

(c) The inclusion $\mathfrak{J}^2 \subseteq I(\mathfrak{J}, \mathfrak{B})$ is obvious. To prove the converse inclusion, assume that $E \in I(\mathfrak{J}, \mathfrak{B})$. Then there is a set $B \in \mathfrak{J} \cap \mathfrak{B}$ such that $E \subseteq B$. Let $B = B_1 \cup B_2$ where $B_1 \in \mathfrak{J}^1$, $B_2 \in \mathfrak{J}^2$. Since $\mathfrak{J}^2$ is Borel, we may assume that $B_2 \in \mathfrak{B}$. Then $B \setminus B_2$ is Borel. Observe that it is countable. Indeed, in the contrary case there is a perfect subset C of $B \setminus B_2$ and then $C \subseteq B_1$ which contradicts (2) . Thus $B \setminus B_2$ is countable and consequently it belongs to $\mathfrak{J}^2$. Hence $B \in \mathfrak{J}^2$. The inclusion $I(\mathfrak{J}, \mathfrak{B}) \subseteq \mathfrak{J}^2$ has been proved. Since $\mathfrak{J}$, $I(\mathfrak{J}, \mathfrak{B})$ are of the same type, therefore $\mathfrak{J}$, $\mathfrak{J}^2$ are of the same type. This ends the proof.

Observe that, by the Alexandroff-Hausdorff theorem, each σ-ideal which consists of totally imperfect sets and contains uncountable sets is non-Borel. Several examples of such σ-ideals are described in [4] (comp. also [2],§ 36).

Now, we shall give some other examples of non-Borel σ-ideals, using Proposition 2.

Example 1. Let $\mathcal{J}^1$ be the σ-ideal of all sets possessing the property (S_0) (see [8]; one of possible definition is : a set E has the property (S_0) if and only if every perfect set contains a perfect set disjoint from E). Then $\mathcal{J}^1$ fulfils (2) and, by assuming the Continuum Hypothesis condition (3) is fulfilled, as well (see [8], 5.3). Observe that $\mathcal{J}^1$ is movable.

Lemma 3. Every perfect set contains $2^{\aleph_0}$ disjoint perfect sets.

Proof. By the Alexandroff-Hausdorff theorem, a perfect set contains a set C homeomorphic with a Cantor set. Let h be a homeomorphism which maps $C \times C$ onto C (comp. [2], p.235). The sets $h(C \times \{t\})$, $t \in C$, just fulfil the assertion.

For any set A denote by $\mathcal{P}(A)$ the family of all subsets of A.

Example 2. Let E be a Bernstein set, i.e. a set such that $D \cap E \neq \emptyset$, $D \setminus E \neq \emptyset$ for each perfect set D (see [5], th. 5.3). By Lemma 3, the sets $D \cap E$, $D \setminus E$ are of power $2^{\aleph_0}$. The set E is totally imperfect, nonmeasurable in the Lebesgue sense and has not the Baire property (see [5], th. 5,4, 5.5). Thus the family $\mathcal{J}^1 = \mathcal{P}(E)$ fulfils conditions (1),(2),(3) of Proposition 2.

Example 3. Let $\mathcal{H}$ be the family of all subsets of $\mathbb{R}$ of power less than $2^{\aleph_0}$. Clearly, $\mathbb{R} \notin \mathcal{H}$, $\mathcal{H}$ is movable and fulfils condition (i). In virtue of the König theorem ([3], p.198), oondition (ii) holds, as well. Thus, by Remark 1, $\mathcal{H}$ is a σ-ideal. Sierpiński constructed in [7] a Bernstein set E such that the symmetric difference $E \Delta (E + x)$ belongs to $\mathcal{H}$ for each $x \in \mathbb{R}$. Let $\mathcal{H}(E) = \mathcal{P}(E) \oplus \mathcal{H}$. Observe that if we put $\mathcal{J}^1 = \mathcal{H}(E)$, then conditions (1),(2),(3) of Proposition 2 will be fulfilled. Indeed, (i),(ii) obviously hold, thus (1) is valid. To verify (2), suppose that there is a perfect set $D \in \mathcal{H}(E)$. Then we have $D \subseteq E \cup H$ for some $H \in \mathcal{H}$, and $E \cap H = \emptyset$ can be assumed. Consequently, $D \setminus E \subseteq H$, which is impossible since $D \setminus E$ is of power $2^{\aleph_0}$ and $H \in \mathcal{H}$. Clearly, the set E quarantees the validity of (3). Next, notice that $\mathcal{H}(E)$ forms a movable σ-ideal. It is a non-Borel σ-ideal since $E \in \mathcal{H}(E)$ and $E \notin I(\mathcal{H}(E), \mathcal{B})$.

Now, our aim will be to demonstrate that if (*) holds, then there is a movable σ-ideal $\mathcal{J}$ of type $(\alpha;\beta)$.

For any nonempty family $\mathcal{C}$ of sets, denote by $\mathcal{C}_\sigma$ (resp. $\mathcal{C}_\delta$), the family of all countable unions (resp. intersections) of sets from $\mathcal{C}$.

Let

$$\mathcal{C}^+ = \{A + x : A \in \mathcal{C}, \ x \in \mathbb{R}\} .$$

$$S^+(\mathcal{C}) = \{A : A : A \subseteq B \text{ for some } B \in (\mathcal{C}^+)_\sigma\} .$$

Proposition 3. Let $\mathcal{C}$ be a family of sets which contains a nonempty set and let $\mathbb{R} \notin (\mathcal{C}^+)_\sigma$. Then $S^+(\mathcal{C})$ is the minimal movable σ-ideal including $\mathcal{C}$. If $\mathcal{C} \subseteq \mathcal{B}$, then the σ-ideal $S^+(\mathcal{C})$ is Borel.

<u>Proof</u>. By the definition of $S^+(\mathcal{C})$, it follows that $S^+(\mathcal{C})$ is a movable family and it fulfils conditions (i), (ii). Thus, by Remark 1, $S^+(\mathcal{C})$ forms a σ-ideal. The inclusion $\mathcal{C} \subseteq S^+(\mathcal{C})$ is obvious. If $\mathcal{J}$ is a movable σ-ideal such that $\mathcal{C} \subseteq \mathcal{J}$, then $(C^+)_\sigma \subseteq \mathcal{J}$ and consequently $S^+(\mathcal{C}) \subseteq \mathcal{J}$. Thus the first assertion holds. If $\mathcal{C} \subseteq \mathcal{B}$, then $(\mathcal{C}^+)_\sigma \subseteq \mathcal{B}$ and so, by the definition of $S^+(\mathcal{C})$, the σ-ideal $S^+(\mathcal{C})$ is Borel. The proof is completed.

In [6] Ruziewicz and Sierpiński constructed a perfect set P such that the set $(P + x) \cap P$ is at most one-point for each $x \neq 0$. Notice that each set

$$(P + x) \cap (P + y) \quad \text{where } x,y \in \mathbb{R} , \ x \neq y ,$$

is also at most one-point.

Let C be a set of measure zero which is included in P and homeomorphic to the Cantor set (see [5], lemma 5.1). Choose pairwise disjoint, perfect sets C_α, C^*_β; $\alpha, \beta < \omega_1$, contained in C (comp. Lemma 3). Since they are included in P; therefore, for all $\alpha, \beta < \omega_1$; $x, y \in \mathbb{R}$, each of the sets

$$(C_\alpha + x) \cap (C_\beta^* + y);$$

$$(C_\alpha + x) \cap (C_\beta + y), \quad (C_\alpha^* + x) \cap (C_\beta^* + y) \text{ for all}$$

for $\alpha \neq \beta$ or $x \neq y$, is at most one-point.

Let $D_0 = D_1 = E_0 = E_1 = \emptyset$ and, for each α, $2 \leq \alpha \leq \omega_1$, let D_α, E_β be such that $D_\alpha \subseteq C_\alpha$, $E_\alpha \subseteq C_\alpha^*$, $D_\alpha \in F_\alpha \setminus G_\alpha$, $E_\alpha \in G_\alpha \setminus F_\alpha$ (see [1]). For each α, $0 < \alpha < \omega_1$, we denote by $T(\alpha)$ the family of all double real-valued sequences $\{t_{n\gamma}\}_{n<\omega, \gamma<\alpha}$. For any $t \in T(\alpha)$, $t = \{t_{n\gamma}\}_{n<\omega, \gamma<\alpha}$ let us denote

$$D(\alpha, t) = \bigcup_{\gamma<\alpha} \bigcup_{n<\omega} (D_\gamma + t_{n\gamma}), \quad E(\alpha, t) = \bigcup_{\gamma<\alpha} \bigcup_{n<\omega} (E_\gamma + t_{n\gamma})$$

__Lemma 4.__ Let $2 \leq \alpha < \omega_1$, $t \in T(\alpha)$. Then

$$D(\alpha, t) \in F_{\alpha-1}, \quad E(\alpha, t) \in G_{\alpha-1} \quad \text{when } \alpha - 1 \text{ exists},$$

and

$$D(\alpha, t), \ E(\alpha, t) \in F_\alpha \cap G_\alpha \quad \text{when } \alpha \text{ is a limit number.}$$

__Proof.__ We shall demonstrate the assertion which deals with $D(\alpha, t)$; the proof concerning $E(\alpha, t)$ is analogous. Notice that $D(2, t) = \emptyset \in F_1$, therefore, in this case, the assertion holds. Now, let $\alpha > 2$. Let $t = \{t_{n\gamma}\}_{n<\omega, \gamma<\alpha}$. Denote

$$C_{n\gamma} = C_\gamma + t_{n\gamma}, \quad D_{n\gamma} = D_\gamma + t_{n\gamma}, \quad D'_{n\gamma} = C_{n\gamma} \setminus D_{n\gamma},$$

$n < \omega$, $\gamma < \alpha$.

From the notations and properties described above it follows that for all k, ξ; $k < \omega$, $\xi < \alpha$, there is a countable set $B_{k\xi}$ included in $C_{k\xi}$ such that

$$C_{k\xi} \setminus D(\alpha, t) = D'_{k\xi} \setminus B_{k\xi}.$$

We then have

$$D(\alpha, t) = \bigcap_{\xi<\alpha} \bigcap_k \Big(\bigcup_{\gamma<\alpha} \bigcup_n (C_{n\gamma} \setminus D'_{k\xi}) \cup B_{k\xi} \Big)$$

which easily implies that $D(\alpha, t) \in \big(\bigcup_{\beta<\alpha} F_\beta\big)_\delta$. om the equality

$$D(\alpha,t) = \bigcup_{\gamma<\alpha} \bigcup_{n} D_{n\gamma}$$

it follows that $D(\alpha,t) \in (\bigcup_{\beta<\alpha} F_\beta)_\sigma$ since $D_{n\gamma} \in F_\gamma$ for all n and for each $\gamma<\alpha$. Thus we have obtained

$$D(\alpha,t) \in (\bigcup_{\beta<\alpha} F_\beta)_\sigma \cap (\bigcup_{\beta<\alpha} F_\beta)_\delta .$$

Assume that $\alpha - 1$ exists. We have

$$(\bigcup_{\beta<\alpha} F_\beta)_\sigma = (F_{\alpha-1})_\sigma = F_{\alpha-1} \quad \text{when } \alpha \text{ is even,}$$

$$(\bigcup_{\beta<\alpha} F_\beta)_\delta = (F_{\alpha-1})_\delta = F_{\alpha-1} \quad \text{when } \alpha \text{ is odd.}$$

Thus $D(\alpha,t) \in F_{\alpha-1}$. If α is a limit number, then

$$(\bigcup_{\beta<\alpha} F_\beta)_\delta = F_\alpha , \quad (\bigcup_{\beta<\alpha} F_\beta)_\sigma = (\bigcup_{\beta<\alpha} G_\beta)_\sigma = G_\alpha .$$

Thus $D(\alpha,t) \in F_\alpha \cap G_\alpha$. The Lemma has been proved.

<u>Lemma 5</u>. If $3 \leq \alpha < \omega_1$, $3 \leq \beta < \omega_1$, $2 \leq \gamma < \alpha$, $2 \leq \xi < \beta$, $s \in T(\alpha)$, $t \in T(\beta)$, then

(a) there is no set $A \in G_\xi$ such that

$D_\xi \subseteq A \subseteq E(\alpha,s) \cup D(\beta,t)$;

(b) there is no set $A \in F_\gamma$ such that

$E_\gamma \subseteq A \subseteq E(\alpha,s) \cup D(\beta,t)$.

<u>Proof</u>. We shall show (a); the proof of (b) is analogous. Suppose that there is a set $A \in G_\xi$ such that

$$D_\xi \subseteq A \subseteq E(\alpha,s) \cup D(\beta,t) .$$

Then, obviously,

$$D_\xi \subseteq C_\xi \cap A.$$

Let $s = \{s_{n\gamma}\}_{n<\omega,\gamma<\alpha}$, $t = \{t_{n\zeta}\}_{n<\omega,\zeta<\alpha}$. In virtue of the construction and notations, the sets $C_\xi \cap (E_\gamma + s_{n\gamma})$, $C_\xi \cap (D_\zeta + t_\zeta)$, $n < \omega$, $\gamma < \alpha$, $\zeta < \beta$, are at most one-point except for the case $\zeta = \xi$, $t_{n\zeta} = 0$ (then $C_\xi \cap (D_\zeta + t_{n\zeta}) = D_\xi$). Hence

$$C_\xi \cap A \subseteq C_\xi \cap (E(\alpha,s) \cup D(\beta,t)) =$$

$$= \bigcup_{\gamma<\alpha}\bigcup_{n} (C_\xi \cap (E_\gamma + s_{n\gamma})) \cup \bigcup_{\zeta<\beta}\bigcup_{n} (C_\xi \cap (D_\zeta + t_{n\zeta})) \subseteq D_\xi \cup B$$

where B is a countable set. We may assume that D_ξ, B are disjoint. Thus

$$D_\xi \subseteq C_\xi \cap A \subseteq D_\xi \cup B ,$$

and so

$$D_\xi = (C_\xi \cap A) \setminus B .$$

Since D_ξ equals the difference of the sets of types G_ξ, F_1, therefore it is of type G_ξ. This contradicts the definition of D_ξ.

Proposition 4. For an arbitrary pair α, β of ordinal numbers such that

$$3 \leq \alpha = \beta \leq \omega_1 \quad \text{or} \quad 3 \leq \alpha + 1 = \beta < \omega_1 \quad \text{or} \quad 3 \leq \beta + 1 = \alpha < \omega_1 ,$$

there is a σ-ideal $\mathfrak{J}(\alpha,\beta)$ which is Borel, movable, of type $(\alpha;\beta)$, included in $\mathcal{L}_1$. Moreover, σ-ideals $\mathfrak{J}(\alpha,\beta)$ can be defined in such a way that if $\alpha \leq \alpha'$ and $\beta \leq \beta'$ then $\mathfrak{J}(\alpha,\beta) \subseteq \mathfrak{J}(\alpha',\beta')$.

Proof. For the α, β fulfilling the assumption, let us put

$$\mathfrak{J}(\alpha,\beta) = S^+ (\{E_\gamma : \gamma < \alpha\} \cup \{D_\gamma : \gamma < \beta\}).$$

Since $D_\gamma \subseteq C_\gamma$, $E_\gamma \subseteq C_\gamma^*$ and C_γ, C_γ^* are closed sets belonging to $\mathcal{L}$, therefore $\mathfrak{J}(\alpha,\beta) \subseteq \mathcal{L}_1$. From Proposition 3 it follows that $\mathfrak{J}(\alpha,\beta)$ is a movable Borel σ-ideal. It is easy to check that if $\alpha \leq \alpha'$ and $\beta \leq \beta'$, then $\mathfrak{J}(\alpha,\beta) \subseteq \mathfrak{J}(\alpha',\beta')$. We have only to show that the σ-ideal $\mathfrak{J} = \mathfrak{J}(\alpha,\beta)$ is of type $(\alpha;\beta)$. At first, assume that $\alpha < \omega_1, \beta < \omega_1$. By the definition of $\mathfrak{J}$, for each $A \in \mathfrak{J}$, there are sequences $s \in T(\alpha)$, $t \in T(\beta)$ such that

(o) $$A \subseteq E(\alpha,s) \cup D(\beta,t) .$$

Of course, the set $B = E(\alpha,s) \cup D(\beta,t)$ belongs to $\mathfrak{J}$. Moreover, by Lemma 4, we have

$$B \in F_\alpha \cap G_\alpha \qquad \text{when} \qquad 3 \leq \alpha = \beta ;$$

$$B \in F_\alpha \qquad \text{when} \qquad 3 \leq \alpha + 1 = \beta \ ;$$

$$B \in G_\beta \qquad \text{when} \qquad 3 \leq \beta + 1 = \alpha \ .$$

Hence $RF(\mathcal{J}) \leq \alpha$, $RG(\mathcal{J}) \leq \beta$. In order to prove the inequalities $RF(\mathcal{J}) \geq \alpha$, $RG(\mathcal{J}) \geq \beta$, observe that if $2 \leq \eta < \alpha$, $2 \leq \xi < \beta$, then $E_\eta \in \mathcal{J} \setminus I(\mathcal{J}, F_\eta)$, $D_\xi \in \mathcal{J} \setminus I(\mathcal{J}, G_\xi)$. For example, we shall show that $E_\eta \in \mathcal{J} \setminus I(\mathcal{J}, F_\eta)$. By the definition of $\mathcal{J}$ we have $E_\eta \in \mathcal{J}$. Suppose that $E_\eta \in I(\mathcal{J}, F_\eta)$. Then there are a set $A \in F_\eta$ and sequences $s \in T(\alpha)$, $t \in T(\beta)$, such that $E_\eta \subseteq A$ and condition (o) holds. This contradicts Lemma 5 (b). Now, assume that $\alpha = \beta = \omega_1$. The inequalities $RF(\mathcal{J}) \leq \omega_1$, $RG(\mathcal{J}) \leq \omega_1$ are evident. The converse inequalities follows from the relations $E_\eta \in \mathcal{J} \setminus I(\mathcal{J}, F_\eta)$, $D_\eta \in \mathcal{J} \setminus I(\mathcal{J}, G_\eta)$ $\eta < \omega_1$. For instance, we shall prove the first of these relatiohs. By the definition of $\mathcal{J}$, we have $E_\eta \in \mathcal{J}$. Suppose that $E_\eta \in I(\mathcal{J}, F_\eta)$. Then there is a set $A \in \mathcal{J} \cap F_\eta$ such that $E_\eta \subseteq A$. By the definition of $\mathcal{J}$, there are a number ξ, $\eta < \xi < \omega_1$, and sequences $s, t \in T(\xi)$ such that $A \subseteq E(\xi, s) \cup D(\xi, t)$. This contradicts Lemma 5 (b).

<u>Theorem 2</u>. Let α, β be an arbitrary pair of ordinal numbers such that $(*)$ holds. Then there are movable σ-ideals $\mathcal{J}(\alpha,\beta)$, $\hat{\mathcal{J}}(\alpha,\beta)$ of type $(\alpha;\beta)$ souh that (α,β) is Borel and included in $\mathcal{L}$, and $\hat{\mathcal{J}}(\alpha,\beta)$ is non-Borel.

<u>Proof</u>. Put $\mathcal{J}(1,2) = \mathcal{L}_1$, $\mathcal{J}(2,1) = \mathcal{L}$, $\mathcal{J}(2,2) = \mathcal{H} \cap \mathcal{L}$ (comp. Proposition 1). Let the remaining σ-ideals be the same as in Proposition 4. Let

$$\hat{\mathcal{J}}(\alpha,\beta) = \mathcal{H}(E) \oplus \mathcal{J}(\alpha,\beta)$$

where $\mathcal{H}(E)$ is the σ-ideal described in Example 3. By Proposition 2, $\hat{\mathcal{J}}(\alpha,\beta)$ is a non-Borel movable σ-ideal of type $(\alpha;\beta)$.

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ABSTRACT

In the paper, for any σ-ideal $\mathfrak{J}$ of subsets of the real line, a type of $\mathfrak{J}$ is defined as a pair $(\alpha;\beta)$ of ordinal numbers such that each Borel set from $\mathfrak{J}$ has supersets from $\mathfrak{J}$ of classes F_α, G_β and α,β are minimal. Some examples are given and a condition necessary and sufficient for a pair $(\alpha;\beta)$ to be a type of a σ-ideal is formulated.

KLASYFIKACJA σ- IDEAŁÓW NA PROSTEJ

Streszczenie

Wprowadza się pewien sposób klasyfikacji σ-ideałów podzbiorów prostej. Jednocześnie autor dokonuje według tego kryterium klasyfikacji kilku znanych przykładów σ-ideałów.