ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ w BYDGOSZCZY
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WELL POSED SOLUTION OF SCHWARZSCHILD INTEGRAL EQUATION AND ITS APPLICATION IN STATISTICAL ASTRONOMY

1. Let us suppose, that it is given an operator equation

$$Ax = y \tag{1}$$

where $x \in X$ and $y \in Y$, and where (X,Y) are Banach spaces. This operator equation is of the first kind.

For this kind of operator equation, in general the problem of solution is not well posed in Hadamard sense. Saying more exactly the problem of solution for equation as (1) is not well posed in Pietrowski's Sobolev's sense.

That means, in Pietrowski's and Sobolev's sense, we need the solution of (1) which must have the property of stability.

We shall omit the well known definition of well posed solution problem in Sobolev's sense, but we shall only remind the condition of stability for the solution of equation as (1).

Def.1. We say, that operator equation (1) has the property of stability on the spaces (X, Y) for given element y, if for every $\varepsilon > 0$ there exists such number $\delta = \delta(\varepsilon) > 0$, that the implication holds :

$$\|y - y_\delta\|_Y < \delta \quad \text{implies} \quad \|x - x_\delta\| < \varepsilon \ , \tag{2}$$

where $y_\delta \in Y$ and $x_\delta \in X$.

But as we have said above, this stability condition in general doesn't hold for the operator equations of the first kind as this one.

However, many important physical and astrophysical problems lead to operator equation (1) .

We can see easily that the Schwarzschild integral equation

$$\int_0^{+\infty} D(r)\varphi[m + 5 - 5 \log r - A(r)] r^2 \, dr = a(m) \ ,$$

where D(r) is stars density and the function φ is luminosity function, and where function A(m) is a derivative of the function N(m) obtained by stars calculating process, is an operator integral equation of the form (1). Therefore all given above remarks hold for this equation.

In this paper we shall investigate the solution of the Schwarzschild integral equation (4) in the modified sense. It means, we shall show, that there exists such a subspace $S \subset X$, for elements of which the stability condition holds.

For this purpose we shall first transform the given Schwarzschild integral equation (4) to the new form.

Taking the Schwarzschild equation (4) we achieve a substitution $\varrho = 5 \log r$. From this substitution we obtain that

$$r = e^{\frac{1}{5 \log e}\varrho} \quad \text{or} \quad r = e^{c\varrho}$$

where

$$c = \frac{1}{5 \log e}$$

Therefore we may write that

$$M = m + 5 - (\varrho + A(e^{c\varrho})) .$$

Using for function $A(e^{c\varphi})$ the approximative value a, we may write that

$$M = m + 5 - (\varrho + a).$$

On the contrary $dr = c\, e^{c\varrho}\, d\varrho$, and the new boundary of integration will be :

$$\varrho_1 = 5 \log 0 = -\infty \;, \quad \varrho_a = 5 \log (+\infty) = +\infty$$

The Schwarzschild integralequation (4) will take the following form

$$\int_{-\infty}^{+\infty} D(e^{c\varrho})\, \varphi\, (m + 5 - (\varrho + a))\, e^{2c\varrho} \cdot c\, e^{c\varrho}\, d\varrho = A(m)$$

or

$$\int_{-\infty}^{+\infty} D(e^{c\varrho})\, c e^{3c\varphi} \varphi(m + 5 - (\varphi + a))\, d\varrho = A(m)$$

Taking once more the substitutions of the form:

$m + 5 = \mu$ and $\varrho + a = R'$, the boundary of integration will be the same, and $d\varrho = dR'$

The new form of the investigated Schwarzschild equation will be of the form:

$$\int_{-\infty}^{+\infty} D\,(e^{c(R'-a)})\,c\,e^{-3ac}\cdot e^{3cR'}(\mu - R')\,dR' = A(\mu - 5),$$

or

$$\int_{-\infty}^{+\infty} D(e^{-ac}\cdot e^{cR'})\;ce^{-ac}\cdot e^{3cR'}(\mu - R')\,dR' = A(\mu - 5),$$

or

$$\int_{-\infty}^{+\infty} D\,(C_1\,e^{cR'})\,C_2 e^{3cR'}\,\varphi(\mu - R')\;dR' = A(\mu - 5)\,.$$

Taking : $D(C_1 e^{cR'})C_2 e^{3cR'} = \Delta_1\,(R')$

and $A(\mu - 5) = \alpha_1(\mu)$ we obtain the convolution form of the Scharzschild integralequation as following

$$\int_{-\infty}^{+\infty} \varphi(\mu - R')\,\Delta_1\,(R')\;dR' = \alpha_1\,(\mu) \tag{5}$$

In the convolution form of the Schwarzschild equation, the $\Delta_1(R')$ is an unknown function, the φ-function as a kernel of the equation is given and the function $\alpha_1\,(\mu)$ is also given.

We can see that the new form (5) of the operator equation is also an operator equation of the first kind.

Therefore for this equation doesn't hold the well posed solution problem in Sobolev sense.

But we may investigate this equation as an operator equation for which the well posed problem of solution is given in Laurent Schwartz sense. We introduce the following

Definition 2. We say that the problem of solution of the convolution form of the equation (5) is well posed in Laurent Schwartz sense if the solution exists in the subspace $S \subset L^1(R)$, it is unique in the space S and if it is stable in the bounded subset S_o of the space $(S; \|\cdot\|_{L^1(R)})$ in sense of the Definition 1. given over.

Here S denotes the space of fast decreasing functions on R which was introduced by Laurent Schwartz in [10].

We shall show that there exists a bounded set $S_o \subset S$ for which elements we shall obtain the well posed problem of solution of the Schwarzschild's integral equation in Laurent

Schwartz sense.

2. For this purpose we shall take an integral equation

$$\int_{\Omega} K(t,\tau)\, x(\tau)\, d\tau = y(t) \quad , \qquad (6)$$

where function $y \in Y$ is given and the solving solution $x \in X$, and where X, Y are Banach spaces such that, by given Kernel $K(t,\tau)$ the operator

$$\tilde{K} = \int_{\Omega} K(t,\tau)(\cdot)\, d\tau \qquad (7)$$

maps the elements $x \in X$ in the elements $y \in Y$, Ω is a given domain of integration variable.

For our purpose the domain Ω will be the real space $\mathbb{R}$, it mean the improper interval $(-\infty , +\infty)$.

Suppose, that the kernel $K(t,\tau)$ will be positive and that the Banach real space $X = Y = L^p(\mathbb{R}, \mu)$, where $p=1$. But in the space $L^p(\mathbb{R}, \mu)$ there exists a cone $\bar{K}$ of positive elements belonging to $L^p(\mathbb{R}, \mu)$. In this case the operator $\tilde{K} : L^p(\mathbb{R}, \mu) \longrightarrow L^p(\mathbb{R}, \mu)$ maps the cone $\bar{K}$ in the form : $\tilde{K}(\bar{K}) \subset \bar{K}$.

Def. 3. We also say that the positive operator $\tilde{K}$ has a monotonical property if the implication :

$$u \leq v \quad \text{implies} \quad \tilde{K}u \leq \tilde{K}v .$$

holds.

Now we shall investigate two integral equation

$$K(t,\tau)\, x(\tau)\, d\tau = y(t) \qquad (8)$$

$$K(t,\tau)\, x(\tau)\, d\tau = y_{\delta}(t) \qquad (9)$$

where the kernel $K_{\delta}(t,\tau)$ approximates the kernel $K(t,\tau)$ in the space $L^p(\ ,\mu)$, where $p = 1$.

Suppose further, that element of space $L^p(\mathbb{R}, \mu)$ y_{δ} approximates the element y of the space $L^p(\mathbb{R}, \mu)$, where $p = 1$.

We suppose that the kernel of this equation fulfils the following conditions: it is measurable in $R \times R$; it is integrable in τ for every $t \in R$ and it is integrable in t for every $\tau \in R$. We assume further, that the integral

$$\varkappa_K(\tau) = \int_R K(t,\tau)\,dt \geq \varkappa_o > 0$$

and that the kernel $K(t,\tau) \in S$, where $S \subset L^1(R)$ is the subspace of fast decreasing functions on R. Further assumptions about the kernel K we shall give below.

Under this assumptions we can write the equality :

$$\int_{\mathbb{R}} K(t,\tau)\,x(\tau)\,d\tau - \int_{\mathbb{R}} K_\delta(t,\tau)\,x_\delta(\tau)\,d\tau = y(t) - y_\delta(t)$$

Now let's modify this equality as follows

$$\int_{\mathbb{R}} K(t,\tau)\;x(\tau)\,d\tau - \int_{\mathbb{R}} K(t,\tau)\,x_\delta(\tau)\,d\tau + \int_{\mathbb{R}} K(t,\tau)\,x_\delta(\tau)\,d\tau -$$

$$\int_R K_\delta(t,\tau)\;x(\tau)\,d\tau = y(t) - y_\delta(t)$$

We obtain that

$$\int_{\mathbb{R}} K(t,\tau)\,(x(\tau) - x_\delta(\tau))\,d\tau = y(t) - y_\delta(t) + \int_R (K_\delta(t,\tau) - K(t,\tau))\cdot$$

$$\cdot\, x_\delta(\tau)\,d\tau$$

Integrating the last equality over the domain $\mathbb{R}$, where $t \in \mathbb{R}$ we obtain the equality

$$\int_{\mathbb{R}}\int_{\mathbb{R}} K(t,\tau)\,\mathrm{sign}(x(\tau) - x_\delta(\tau))\,|x(\tau) - x_\delta(\tau)|\,d\tau\,dt =$$

$$= \int_{\mathbb{R}} (y(t) - y_\delta(t))\,dt + \int_R\int_R (K_\delta(t,\tau) - K(t,\tau))\,x_\delta(\tau)\,d\tau\,dt$$

Supposing that the Fubini theorem holds we obtain the equality

$$\int_R \Big(\int_R K(t,\tau)\;dt\Big)\,(\mathrm{dign}(x(\tau) - x_\delta(\tau))\,|\,x(\tau) - x_\delta(\tau)|\,d\tau =$$

$$= \int_R (y(t) - y_\delta(t))\,dt + \int_R \Big(\int_R (K_\delta(t,\tau) - K(t,\tau))\,dt\Big)\,x_\delta(\tau)\,d\tau$$

From this equality we easily obtain the inequality

$$\Big|\int_R \mathrm{sign}(x(\tau) - x_\delta(\tau))\,|\,x(\tau) - x_\delta(\tau)|\,\varkappa_K(\tau)\,d\tau \leq$$

$$\leq \int_R |y(\tau) - y_\delta(\tau)|\,d\tau + \int_R k(\tau)\,|x_\delta(\tau)|\,d\tau\,, \qquad (10)$$

where we have designed :

$$\varkappa_K(\tau) = \int_{\mathbb{R}} K(t,\tau)\quad dt \quad \text{for } \tau \in \mathbb{R} \quad \text{and}$$

$$k(\tau) = \int_R |K_\delta(t,\tau) - K(t,\tau)|\,dt \qquad \text{for } \tau \in \mathbb{R}\,.$$

In addition, if we are able to approximate the kernel $K(t,\tau)$ by $K_\delta(t,\tau)$ as well, that the integral $k(\tau)$ will be independent of $\tau \in R$, and the value of $k(\tau) = k_o$ = constant, will be sufficiently small that is if for $\tau \in \mathbb{R}$ is $k(\tau) < \eta$, we obtain instead of the inequality (10) the inequality,

$$\left|\int_{\mathbb{R}} \operatorname{sign}(x(\tau) - x_\delta(\tau))|x(\tau) - x_\delta(\tau)|\,d\mu\right| \leq$$
$$\leq \int_R |y(\tau) - y_\delta(\tau)|\,d\tau + \int_R \eta\,|x_\delta(\tau)|\,d\tau \qquad (11)$$

Where $d\mu = \varkappa_K(\tau)\,d\tau$, that means, that the measure μ has a density $\varkappa_K(\tau)$, $\tau \in \mathbb{R}$ and η is a sufficiently small positive number.

Suppose, that also the integral $\varkappa_K(\tau)$ is independent of τ on domain $\mathbb{R}$. It means that the measure μ is of constant density $\varkappa_K = \varkappa_0 > 0$, we obtain instead of inequality (11) the inequality

$$\left|\int_R \operatorname{sign}(x(\tau) - x_\delta(\tau))|x(\tau) - x_\delta(\tau)|\varkappa_0\,d\tau\right| \leq$$
$$\leq \int_{\mathbb{R}} |y(\tau) - y_\delta(\tau)|\,d\tau + \eta\int_R |x_\delta(\tau)|\,d\tau$$

or
$$\left|\int_R \operatorname{sign}(x(\tau) - x_\delta(\tau))|x(\tau) - x_\delta(\tau)|\,d\tau\right| \leq$$
$$\leq \frac{1}{\varkappa_0}\|y - y_\delta\|_{L^1(R)} + \frac{\eta}{\varkappa_0}\|x_\delta\|_{L^1(R)} \qquad (12)$$

Now, if the approximation of y by y_δ in the space $L^1(\mathbb{R})$ is as well, that $\|y - y_\delta\|_{L^1(R)} < \varkappa_0 \eta_0$, where $\eta_0 > 0$ is a small number, and if we assume that the solution x_δ is bounded in the space $L^1(\mathbb{R})$ that means, that the solution x_δ belongs to the ball $B = \{x_\delta : \|x_\delta\|_{L^1(R)} \leq \varkappa_0 \eta_0\}$ in the space $L^1(R)$ then we obtain that the following inequality holds

$$\left|\int_R \operatorname{sign}(x(\tau) - x_\delta(\tau))|x(\tau) - x_\delta(\tau)|\,d\tau\right| \leq \eta_0 + b_0\eta \leq$$
$$\leq \eta_0 + b_0 \frac{1}{b_0}\eta_1 = \eta_0 + \eta_1 \leq \varepsilon \qquad (13)$$

The above investigation may be without sense, if there does not exist such a kernel for which conditions required by us may be not satisfying.

But the class of such kernels $K(t,\tau)$ is not empty.

The kernel

$$K(t,\tau) = E\,e^{-\omega(t-\tau)^2} \qquad t,\tau \in R \qquad (14)$$

as we easy show fulfil our required conditions.

The kernel (14) is positively defined on the space $\mathbb{R}$ if $E > 0$. The integral

$$\varkappa_K(\tau) = \int_R E\, e^{-\omega(t-\tau)^2}\, dt = E\int_R e^{-\omega(t-\tau)^2}\, dt =$$
$$= E\int_R e^{-\omega u^2}\, du = E\sqrt{\frac{\pi}{\omega}}$$
, it means, that for the kernel (14) the integral $\varkappa_K(\tau)$ is independent of τ belonging to the interval $(-\infty, +\infty)$.

If we give two such kernels

$$K(t,\tau) = E\, e^{-\omega(t-\tau)^2} \quad \text{and}$$
$$K_\delta(t,\tau) = E\, e^{-\omega_\delta(t-\tau)^2} \tag{15}$$

then we can see, that $k(\tau) =$

$$= \int_R |K_\delta(t,\tau) - K(t,\tau)|\, d\tau = \int_R \left| E_\delta\, e^{-\omega_\delta(t-\tau)^2} - E e^{-\omega(t-\tau)^2}\right| d\tau \le$$
$$\le \int_R \left(\left| E_\delta e^{-\omega_\delta(t-\tau)^2}\right| + \left| E e^{-\omega(t-\tau)^2}\right|\right) d\tau = \int_R E_\delta e^{-\omega_\delta(t-\tau)^2}\, d\tau +$$
$$+ \int_R E e^{-\omega(t-\tau)^2}\, d\tau = E_\delta\sqrt{\frac{\pi}{\omega_\delta}} + E\sqrt{\frac{\pi}{\omega}} \le$$
$$\le \max(E, E_\delta)\left(\max\left(\sqrt{\frac{\pi}{\omega_\delta}}\, ;\, \sqrt{\frac{\pi}{\omega}}\right)\right) = E_0\sqrt{\frac{\pi}{\omega_0}} = k_0$$

But we can see that two positive numbers E_δ and ω_δ may be always chosen so that the product $E_0\sqrt{\frac{\pi}{\omega_0}}$ will be sufficiently small.

Indeed, we have

$$I = \int_R \left| E_\delta e^{-\omega_\delta(t-\tau)^2} - E e^{-\omega(t-\tau)^2}\right| d\tau = \int_R \left| E_\delta e^{-\omega_\delta u^2} - E e^{-\omega u^2}\right| du =$$
$$= \int_R E e^{-\omega u^2}\left| \frac{E_\delta}{E} e^{-(\omega_\delta-\omega)u^2} - 1\right| du .$$

But, taking $\eta > 0$ and $R_0 > 0$ such that

$$ME \int_{|u|>R_0} e^{-\omega u^2}\, du < \frac{\eta}{4} \quad \text{and} \quad \frac{E_\delta}{E} < 1 + \frac{\eta}{8R_0},$$

we obtain farther that

$$I \le \int_{|u|>R_0} E e^{-\omega u^2} M du + \int_{-R_0}^{R_0} e^{-\omega u^2}\left| (1+\eta) e^{-(\omega_\delta-\omega)u^2} - 1\right| du \le$$
$$\le \frac{\eta}{4} + \int_{-R_0}^{R_0} \left| \left(1 + \frac{\eta}{8R_0}\right) e^{-(\omega_\delta-\omega)u^2} - 1\right| du \le \eta .$$

The last inequality will be true since we have for sufficiently small $\delta > 0$ the implication: $\omega_\delta \longrightarrow 0$ if $\delta \longrightarrow 0$ and we can see easy that:

$$f_\delta(u) = \left| \left(1 + \frac{\eta}{8R_0}\right) e^{-(\omega_\delta-\omega)u^2} - 1\right| \rightrightarrows \frac{\eta}{8R_0} ,$$

if $\delta \to 0$ for $u \in \langle -R_o, R_o \rangle$.
For we have the following inequality

$$\sup_{|u|\le R_o} f_\delta(u) = \max\left(\frac{\eta}{8R_o}, \left|\left(1 + \frac{\eta}{8R_o}\right) e^{-(\omega_\delta - \omega)u^2} - 1\right|\right) \le \frac{\eta}{8R_o} + \cdots$$

Since

$$f'_\delta(u) = \left(1 + \frac{\eta}{8R_o}\right)\left(-2(\omega_\delta - \omega)\, u e^{-(\omega_\delta - \omega)u^2}\right) = 0, \text{ gives } u = 0.$$

Then, for

$$f_\delta(u) = \left(1 + \frac{\eta}{8R_o}\right) e^{-(\omega_\delta - \omega)R_o^2} - 1$$

we obtain that

$$I \le \frac{\eta}{4} + \int_{-R_o}^{R_o} \frac{\eta}{8R_o}\, du + \int_{-R_o}^{R_o} \left|\left(1 + \frac{\eta}{8R_o}\right) e^{-(\omega_\delta - \omega) R_o^2} - 1\right| du \le$$

$$\le \frac{\eta}{4} + \frac{\eta}{4} + 2R_o \left|\left(1 + \frac{\eta}{8R_o}\right) e^{-(\omega_\delta - \omega) R_o^2} - 1\right| \le \frac{\eta}{4} + \frac{\eta}{4} + \frac{\eta}{2} = \eta .$$

It means, for kernel as (14) and (15) it is always possible to obtain the inequality

$$k(\tau) = \int_{\mathbb{R}} |K_\delta(t,\tau) - K(t,\tau)|\, d\tau < \eta .$$

Therefore we may formulate the following

Lemma 1. For the integral equation (6) with the kernel (14) and (15) in the space $X = Y = L^1(\mathbb{R})$ the inequality (13) is true if the solution x_δ belongs to a ball in the space $L^1(\mathbb{R})$.

From this Lemma 1 we have the Corollary 1. If for the solution x_δ of integral equation (9_δ) the Lemma 1 is true, than the solution x_δ is stable in sense:

$$\left|\int_{R} \operatorname{sign}(x(\tau) - x_\delta(\tau))\, |x(\tau) - x_\delta(\tau)|\, d\tau\right| < \varepsilon .$$

But in a special case, if for solutions x_δ equation (9_δ) is always fulfilled, the inequality

$$x_\delta < x$$

which is equivalent with the inequality $\|x_\delta\| \le \|x\|$, we obtain simple form of stability in the norm sense in the space $L^1(R)$, that is

$$\int_{\mathbb{R}} |(x(\tau) - x_\delta(\tau))|\, d\varepsilon < \varepsilon .$$

But in this special case, we may write the inequality (12) in the form

$$\|x - x_\gamma\|_{L^1(R)} \leq \frac{1}{\varkappa_0} \|y - y_\gamma\|_{L^1(R)} + \frac{\gamma}{\varkappa_0} \|x_\gamma\|_{L^1(R)} < \varepsilon \quad (16)$$

This result we may express in the

Lemma 2. The solution x_γ of (9) belonging to the set $S_0 = B \cap S$ is stable in sense of the Definition 1, that is

$$\|y - y_\gamma\| < \eta \quad \text{implies} \quad \|x - x_\gamma\| < \varepsilon$$

holds, if $x, x_\gamma \in S_0$ and $y, y_\gamma \in S$.

In conclusion we have the

Theorem 1. Let us suppose that the kernel $K(t,\tau)$ of the equation (6) fulfils the following conditions:

1^0 It is measurable in $R \times R$

2^0 It is integrable in τ for every $t \in R$ and it is integrable in t for every $\tau \in R$.

3^0 The integral

$$\varkappa_K(\tau) = \int_R K(t,\tau) \, dt \geqslant \varkappa_0 > 0$$

4^0 The kernel $K(t,\tau) \in S$, where $S \subset L^1(R)$ is the subspace of fast decreasing functions on R and it is positive and has a monotonical property.

5^0 It is given the ball

$$B = \{x_\gamma : \|x_\gamma\|_{L^1(R)} \leq \varkappa_0 \, b_0\}$$

where the constant $\varkappa_0$ is defined by the integral in condition 3^0 and where the constant b_0 is a real number. Then the problem of the solution of the Schwarzschild's integral equation is well posed in Laurent Schwartz sense in the set $S_0 = B \cap S$.

5. Now we return to the Schwarzschild's integral equation in the convolution form (5) .

By application of equation (5) to the stars calculating process the kernel φ may be approximated by the exponential function

$$\varphi(M) = E\, e^{-a\,(M - M_o)^2} \tag{17}$$

We may also approximate the function $\alpha_1(m)$ on the right side of equation (5) by an exponential function

$$\alpha_\delta(\mu)\ E_\delta\, e^{-b(\mu - \mu_o)^2} \quad . \tag{18}$$

Now the Scharzschild's integral equation in the convolution form will be as following

$$\int_R E\, e^{-a(\mu-R-\mu_o)^2} \Delta_1(R)dR = E_\delta\, e^{-b(\mu-\mu_o)^2} \; .$$

Putting $\mu - \mu_o = t$, where $\mu = t + \mu_o$ we may write

$$\int_R E\, e^{-a(t-R+\mu_o)^2}\ \Delta_1(R)\ dR = E_\delta\, e^{-bt^2}$$

and putting $R - \mu_o = \tau$; $dR = d\tau$, we have

$$\int_R E\, e^{-a(t-\tau)^2}\ \Delta_1(\tau + \mu_o)\ d\tau = E_\delta\ e^{-bt^2} \quad .$$

Designet $\Delta_1(\tau + \mu_o)$ by $\Delta_o(\tau)$ we obtain a simple form of the Schwarzschild integral equation

$$\int_R E\, e^{-a\,(t-\tau)^2}\ \Delta_o(\tau)\ d\tau = E_\delta\, e^{-bt^2} \tag{19}$$

where the function $\Delta_o(\tau)$ is unknown,

Obviously,. the kernel φ and given over function on the right side of the equation (19) both belong to the space $S \subset L^1(R)$. It is useful to solve the integral equation (19) with help of the Fourier transformation $F(f) = \int_R e^{-i\omega\tau}\ f(\tau)\,d\tau$ on S. Using the Fourier transformation F to both side of equation (19) we obtain

$$F(\int_R E\, e^{-a(t-\tau)^2} \Delta_o(\tau) = F\,(E_\delta\, e^{-bt^2}) \; .$$

The Fourier transformed equation (19) gives an equation

$$E\sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4a}}\ F(\Delta_o) = E_\delta\sqrt{\frac{\pi}{b}}\ e^{-\frac{\omega^2}{4b}} \tag{20}$$

from whence

$$F(\Delta_o) = \frac{E_\delta}{E}\sqrt{\frac{a}{b}}\ e^{-\frac{1}{4}\left(\frac{1}{b} - \frac{1}{a}\right)\omega^2} \quad ,$$

and after easy modification

$$F(\Delta_o) = \frac{E_\delta}{E}\sqrt{\frac{a}{b}}\sqrt{\frac{\frac{ab}{a-b}}{\pi}}\ \sqrt{\frac{\pi}{\frac{ab}{a-b}}}\ e^{-\frac{\omega^2}{4\frac{ab}{a-b}}} \tag{21}$$

Using to the Fourier image of Δ_o the inverse Fourier transformation F^{-1} we obtain for the researched solution the function

$$\Delta_o(t) = \frac{E_\delta}{E} \frac{a}{\sqrt{\pi(a-b)}} e^{-\frac{ab}{a-b}t^2} \qquad (22)$$

Of course, the researched solution $\Delta_o(t)$ given by formula(22) is bounded in the space $S \subset L^1(R)$ and belongs to the ball

$$S_o = \left\{ x_\delta : \int_R |x_\delta(\tau)| d\tau \le B_o \right\} = \left\{ x_\delta : \|x_\delta\|_{L^1(R)} \le B_o \right\} \qquad (23)$$

where

$$B_o = \frac{E_\delta}{E} \frac{a}{\sqrt{\pi(a-b)}} \sqrt{\frac{\pi}{\frac{ab}{a-b}}} = \frac{E_\delta}{E} \sqrt{\frac{a}{b}} \qquad (24)$$

Before we pass to an ilustrative example of application the Schwarzschild's integral equation to founding the density function $D(r)$, we present shortly the used method in the stars countings problem with help of which we will estimate the parameter of the kernel φ and the α_δ on the right side of Schwarzschild's equation.

4. The classical method of determination of the distribution of stars and interstellar dust from the magnitudes, colour indices and spectral types of stars was described by R.J. Trumpler and H.F. Weaver in their monograph. This method enables to determine the interstellar extinction and the density of stars $D(r)$ of different spectral class and luminosity groups.

In our method, the function $\alpha_1(\mu)$- equation (5) were obtained from stellar counts in the Sagitta field (see C. Iwaniszewska, S. Grudzińska).

We have for $dm = 0^m5$

$$A(m)\frac{d\,N(m)}{dm} = N\left(m + \frac{1}{4}\right) - N\left(m - \frac{1}{4}\right) \qquad (25)$$

where $N(m)$ is the number of stars of apparent magnitude m_{pg} (photographic magnitude).

The results of stellar counts for seven regions of Sagitta field $4^\circ x 4^\circ$ only for 167 stars of spectral type A0 are following :

A (9) = 6	A (11,5) = 42
A (9,5) = 19	A (12) = 22
A (10) = 11	A (12,5) = 15
A (10,5) = 29	A (13) = 2
A (11) = 30	A (13,5) = 1

The run of the A(m) with m in the solid angle of one square degree is presented on Fig. 1.

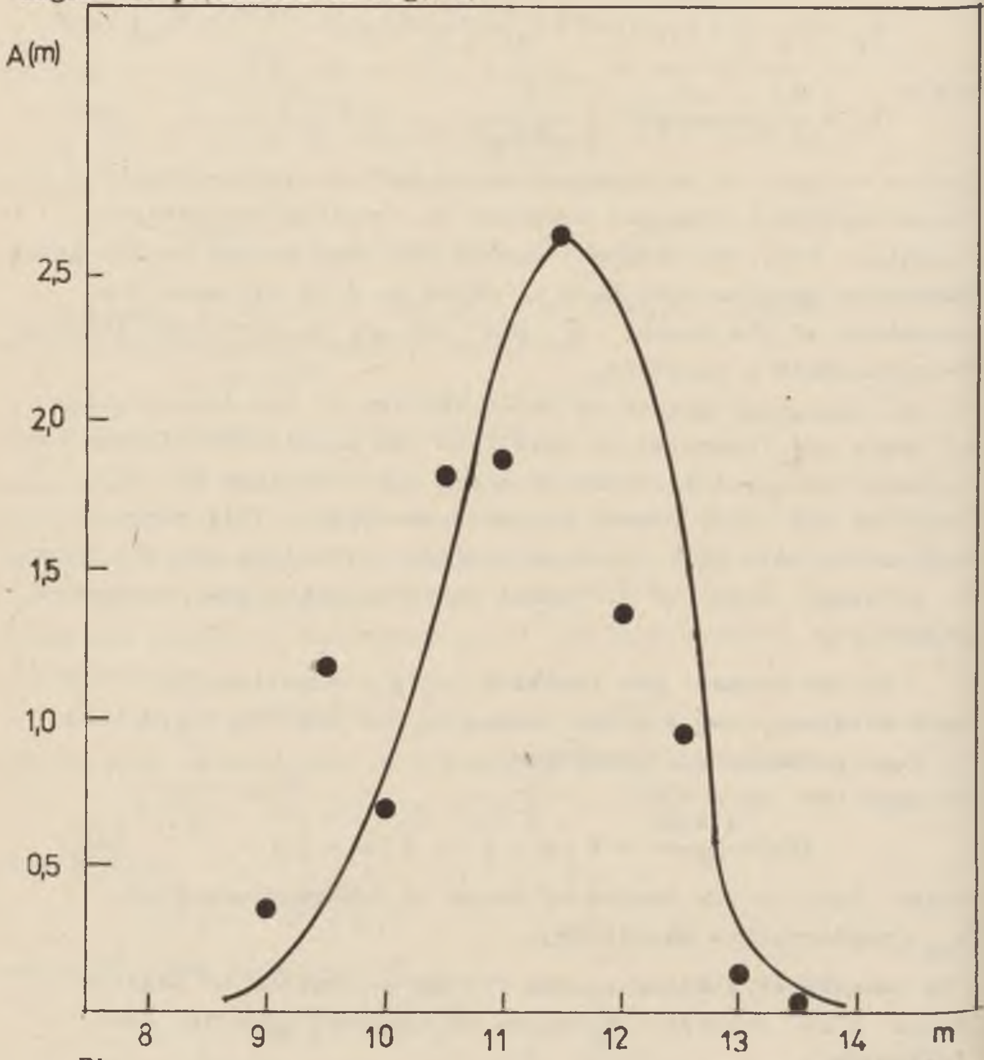

Fig. 1 Stellar counts for stars of the spectral type A0 in the Sagitta field

The obtained A(m) curve may be calculated in the form

$$A(m) = 2,6 \; e^{-0,6\,(m - 11,5)^2} \qquad (26)$$

The luminosity function $\varphi(M)$ for the same spectral type stars A0 was calculated. We assume the absolute magnitude values for A0 stars from Allens's Tables. The standard dispersion was taken from the McCuskey's paper and R. Ampel. We obtained

$$\varphi(M) = 0,5 \; e^{-0,78(M - 0,4)^2} \qquad (27)$$

5. For the kernel φ given by formula (27) we have E = 0,5, a = 0,78, M_o = 0,4 and for the function α_δ on the right side given by formula (26) we have E = 2,6, b = 0,6 , μ_o = 11,5 . The Schwarzschild's integral equation

$$\int_R 0,5 \;\; e^{-7,8\,(t-\tau)^2} \Delta_o(\tau)\, d\tau = 2,6 \; e^{-0,6\; t^2}$$

has a solution given by formula (22)

$$\Delta_o(t) = \frac{2,60}{0,50} \; \frac{0,78}{\sqrt{\pi\,(0,78 - 0,60)}} \;\; e^{-\frac{0,78 \;\; 0,60}{0,78 - 0,60}\; t^2}$$

After performing the rule of calculation we obtain the function

$$\Delta_o(t) = 5,204 \;\; e^{-0,26\; t^2}$$

Now we must come back to the density function D(r). We have

$$\Delta_o(\tau)\, d\tau = \Delta_1(\tau + \mu_o)\, d\tau = \Delta_1(R) dR =$$

$$= D(C_1\; e^{cR})\, C_2\; e^{3cR}\; dR = D\,(e^{c(R-a)})\; ce^{3c(R-a)} dR =$$

$$= D\,(e^{c\varrho})\, c\; e^{3c\varrho} d\varrho = D\,(e^{c\varrho}) c\; e^{c\varrho}\; e^{2c\varrho}\; d\varrho =$$

$$= D(e^{c\varrho})\,(e^{c\varrho})^2 c\; e^{c\varrho}\, d\varrho = D(r)\; r^2 dr$$

$$\Delta_o(\tau) = 5,204 \;\; e^{-0,26\tau^2} d\tau = 5,204 \;\; e^{-0,26\; R^2} dR =$$

$$= 5,204\; e^{-0,26\; R^2}\; dR = D(r)\; r^2 dr$$

$$D(r) = 5,204 \;\; e^{-0,26\; R^2} \frac{dR}{r^2 dr} = 5,204\; e^{-0,26(\varrho + a)^2} \frac{d(\varrho + a)}{r^2 dr}$$

$$5{,}204\ e^{-0{,}26(5\log r + a)^2}\ \frac{d(5\log r + a)}{r^2 dr}$$

$$5{,}204\ e^{-0{,}26(5\log r + a)^2}\ \frac{1}{r^2}\cdot\frac{5}{r}\log e =$$

$$5{,}204\ e^{-0{,}26(5\log r + a)^2}\cdot\frac{5\log e}{r^3}$$

$$D(r) = 26{,}02\ \frac{\log e}{r^3}\ e^{-0{,}26\,(5\log r + a)^2}$$

In our ilustrative example of application the Schwarzschild's equation to finding the density $D(r)$ we have found for stars of A0 spectral type

$$D_{A0}(r) = \frac{11{,}3}{r^3}\ e^{-0{,}26\,(5\log r + a)^2}$$

Remarks. Further investigation of the density D_{A0} and examples of application the Schwarzschild's integral equation to finding the density $D(r)$ in other stellar fields will be published in the next papers . The stability of the used approximation method will be considered also.

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O POPRAWNYM ROZWIAZANIU RÓWNANIA CAŁKOWEGO SCHWARZSCHILDA I JEGO ZASTOSOWANIACH W ASTRONOMII STATYSTYCZNEJ

Streszczenie

W tym artykule formułuje się w oparciu o pojęcie poprawnego rozwiązania w sensie L. Schwartza warunki poprawnego rozwiązania równania całkowego Schwarzschilda. Ustala się klasę jąder. Pokazuje się, że klasa ta nie jest pusta przy jądrze typu krzywej gaussowskiej.

Otrzymane rezultaty stosuje się do aproksymacji zliczeń gwiazd. Dalsze badania i rozwinięcia tej tematyki będą kontynuowane w następnych artykułach.