ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ w BYDGOSZCZY
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ON SOME GEOMETRICAL CHARACTERIZATION OF SINGULAR NORMED MEASURES

Let X be a vector space (real or complex) and let K be a subset of X having at least two points. We shall say that two different points $p_1, p_2 \in K$ are antipodal in K (or simply antipodal) if and only if for every $x_1, x_2 \in K$ and for every real number t the equality $t(p_1 - p_2) = x_1 - x_2$ implies $|t| \leq 1$.

It is easy to prove that for every pair x_1, x_2 of different points belonging to the compact set K in Hausdorff topologioal vector space X there exists a pair of antipodal points $p_1, p_2 \in K$ and a real number t, $|t| \leq 1$ such that $t(p_1 - p_2) = x_1 - x_2$.

Let $(X, \mathcal{A})$ be any measurable space and μ, ν nonnegative measures defined on this space and normed by the condition $\mu(X) = \nu(X) = 1$. Using the Jordan decomposition theorem we can prove the next

THEOREM 1. Two normed measures μ, ν defined on X are antipodal if and only if $|\mu - \nu|(X) = 2$, when $|\mu - \nu|(X)$ means the total variation of a signed measures $\mu - \nu$ on X.

From this theorem and Hahn decomposition theorem we can obtain a simple geometrical characterization of antipodal measures on X.

THEOREM 2. Two normed measures μ, ν defined on X are antipodal if and only if they are singular.

Let us consider the class $\{(X_\gamma, \mathcal{A}_\gamma)\}_{\gamma \in \Gamma}$ of measurable spaces and the families $\{\mu_\gamma\}_{\gamma \in \Gamma}, \{\nu_\gamma\}_{\gamma \in \Gamma}$ of normed measures defined on X. Put $\mu = \bigotimes_{\gamma \in \Gamma} \mu_\gamma$, $\nu = \bigotimes_{\gamma \in \Gamma} \nu_\gamma$. Using theorem 2 it is easy to prove

THEOREM 3. If there exists $\gamma_o \in \Gamma$ such that $\mu_{\gamma_o}, \nu_{\gamma_o}$ are antipodal then the product measures μ, ν are antipodal.

If Γ is a finite set then the above theorem can be reversed. Using the Lebesgue-Radon-Nikodym theorem we can prove

THEOREM 4. The measures $\mu = \bigotimes_{k=1}^{n} \mu_k$, $\nu = \bigotimes_{k=1}^{n} \nu_k$ defined on a product$\left(\mathbb{P}_{k=1}^{n} X_k, \mathbb{P}_{k=1}^{n} \mathcal{A}_k\right)$ of measurable spaces and normed by the condition $\mu_k(X_k) = \nu_k(X_k) = 1$, $k=1,2,\dots n$, are antipodal if and only if there exists a natural number k_o, $1 \le k_o \le n$ such that the measures μ_{k_o}, ν_{k_o} are antipodal.

We shall construct the example showing that the theorem 3 can not be reversed if μ, ν are the product measures on the product of infinitely many measurable spaces. Suppose that $X_n = \langle 0,1 \rangle$, $\mathcal{A}_n$ are Borel subsets of X_n. Put for all $n \in N$ and for each Borel subset $E \subset \langle 0,1 \rangle$

$$\nu_n(E) = \begin{cases} 1 & \frac{1}{n} \in E \\ 0 & \frac{1}{n} \notin E \end{cases}, \qquad \mu_n(E) = \begin{cases} \frac{k}{n^2} & \text{card } E \cap \left\{\frac{1}{n^2}, \frac{2}{n^2}, \dots 1\right\} = k \\ 0 & E \cap \left\{\frac{1}{n^2}, \frac{2}{n^2}, \dots 1\right\} = \emptyset \end{cases}$$

It easy to see that $\nu_n \ll \mu_n$. Let $\mu = \bigotimes_{n=1}^{\infty} \mu_n$, $\nu_n = \bigotimes_{n=1}^{\infty} \nu_n$

Sign by B an arbitrary measurable subset of the product σ-algebra $\mathbb{P}_{n=1}^{\infty} \mathcal{A}_n$. Then we have

$$\nu(B) = \begin{cases} 1 & (1, \frac{1}{2}, \frac{1}{3}, \dots) \in B \\ 0 & (1, \frac{1}{2}, \frac{1}{3}, \dots) \notin B \end{cases}$$

and $\mu\left(\left\{(1, \frac{1}{2}, \frac{1}{3}, \dots)\right\}\right) = \prod_{n=1}^{\infty} \mu_n \left(\left\{\frac{1}{n}\right\}\right) = \prod_{n=1}^{\infty} \frac{1}{n} = 0.$

So if we put $A = \left\{(1, \frac{1}{2}, \frac{1}{3}, \dots)\right\}$ then $\mu(A) = \nu\left(\mathbb{P}_{n=1}^{\infty} X_n \setminus A\right) = 0$, what means that μ, ν are antipodal.