ON PREPONDERANTLY EQUICONTINUOUS COLLECTIONS OF TRANSFORMATIONS

The purpose of this article is to show that a problem posed by Z. Grande in [10] has an affirmative answer, even in a more general setting than it is required in [10]. At the same time we give a solution of the question stated at the end of [13] and we prove some related theorems. In what follows \((X, d_X), (Y, d_Y), (Z, d_Z)\) denote three separable, complete metric spaces, the first of which is equipped with a positive Borel measure \(m\) such that \(m(K(x^0, r)) < +\infty\) and \(\inf \{m(K(x^0, r)) : x^0 \in X\} > 0\) for all \(r > 0\) where \(K(x^0, r) := \{x \in X : d_X(x^0, x) < r\}\) is an open ball centered at \(x^0 \in X\) and with radius \(r\).

\(R\) denotes as usually the real line endowed with the euclidean distance. Given an arbitrary set \(F\) we denote the space of all bounded transformations on \(F\) whose target space is \(Z\) by \(B(F, Z)\). This space is completely metrized by the uniform metric \(D\) defined by:

\[
D(h_1, h_2) := \sup \{ d_Z(h_1(f), h_2(f)) : f \in F \}.
\]

By \(Z^X\) we denote the space of all transformations defined on \(X\) and with values in \(Z\).

**DEFINITION 1.** A family \(F\) of \(m\)-measurable transformat-
ions $f : X \to Z$ is said to be preponderantly equicontinuous (cf. [10], p. 22) if there is a multifunction $E$ from $X$ into the hyperspace of nonempty $m$-measurable subsets of $X$ and a positive real-valued function $\delta : X \to \mathbb{R}_+$ such that for all $x^0 \in X$ we have:

(a) $x^0 \in E(x) \cap \text{der } E(x^0)$, where $\text{der } E(x)$ denotes the set of all accumulation points of $E(x)$;

(b) the ratio $\frac{m(U(x^0) \cap E(x^0))}{m(U(x^0))}$ is greater than $1/2$ whenever $U(x^0)$ is an open neighbourhood of $x^0$ whose diameter $\text{diam } U(x^0) := \sup d_X(U(x^0) \times U(x^0)) / \delta(x)$

(c) the restrictions $\{f \mid E(x^0) : f \in F\}$ create a family equicontinuous at $x^0$. This means that:

(1) $\bigwedge \bigvee \bigwedge \bigwedge [x \in E(x^0) \cap K(x^0, \rho)] \Rightarrow \rho > 0 \Rightarrow f \in F \Rightarrow \forall x \in X$

\[ \Rightarrow d_Z(f(x^0), f(x)) \leq \varepsilon \]

If $F = \{f\}$ consists of a single transformation $f : X \to Z$, then the above definition 1 reduces to preponderant continuity of $f$ (cf. [5], [12], [18], [22]). Note that there is no topology $T$ on $X$ for which preponderantly continuous functions were exactly $T$-continuous. This follows from the fact that for two distinct preponderantly continuous at $x \in X$ functions $f, g : X \to \mathbb{R}$ the measure $m(E^f(x) \cap E^g(x))$ may be arbitrarily small in each neighbourhood of $x$ and thus $f + g$ may fail to be preponderantly continuous at $x \in X$. In [13] Z. Grande has been introduced the following definition:

**DEFINITION 2.** A family $F \subseteq Z^X$ of transformations $f : X \to Z$ fulfills the property $A_2$ if for each nonvoid closed
subset $K$ of $X$ there is a point $x^0 \in K$ such that the restrictions $\{f|_K : f \in F\} \subset Z^K$ form an equicontinuous at $x^0$ collection of transformations ((1) with $K$ instead of $E(x^0)$).

We shall shortly write $F \in A_2$ in that situation.

A more general notion has been investigated by Biagio Ricceri (Rocky Mountain J. of Math., vol. 14, no 3 , 1984, pp. 503-517). Under his terminology the functions from the family $F \subset A_2$ are equibelonging to the first Baire class. If $F$ consists of a single transformation $f$, then $\{f\} \subset A_2$ simply means that $f$ is of the first Baire class [19].

If $f : X \times Y \to Z$, we shall call a family of transformations $f_x : Y \to Z$, $x \in X$ defined by $f_x(y) := f(x,y)$, the $X$-sections of $f$. The $Y$-sections are defined similarly by $f^y(x) := f(x,y)$.

Numerous papers were devoted to the conditions guaranteeing the Borel measurability of a transformation, expressed in terms of its sectionwise properties cf. a chart in [17], p. 169.

In particular [13] essentially contains the following deep theorem:

**THEOREM 0.** (cf. [13]). If $g : X \times Y \to Z$ is a transformation such that:

(d) $\{g^y : y \in Y\} \subset A_2$ and

(e) all sections $g_x : Y \to Z$, $x \in X$ belong to the Baire class $\alpha$, $0 < \alpha < \Omega$, then $g$ also belongs to the Baire class $\alpha$.

In case $X = Y = Z = R$ this is exactly the theorem 6 from [13]. Although the possibility of generalizing the domain is not mentioned in Remark 3 on p. 125 in [13], but this is evident by the penetrating inspection of the original
proof. The generalization of the range space $Z$ is permitted, as it follows from the equality $g^{-1}(K(x, y)) = \{ (x, y) \in X \times Y : d_Z(z, g(x, y)) < r, \forall z \in P_0(X \times Y) \}$ by virtue of the fact that for all $m \in Z$ a real-valued function $(x, y) \mapsto g_Z(x, y) := d_Z(z, g(x, y))$ fulfills assumptions (d) and (e) and each open set $V$ in $Z$ is a countable union of open balls in the presence of the separability of $Z$, cf. [15].

**Lemma 1.** If $0 \leq d_X(x_1^i, x_2^i) < \tau - 1$ and $(< x_1^i, x_2^i >)$ and

\[(2) \quad U(x_1^i, x_2^i) := \bigcap_{i=1}^2 K(x_1^i, \tau - 1 \delta(x_1^i)), \text{ then the intersection } \cap_{i=1}^2 U(x_1^i, x_2^i) \text{ is nonempty.} \]

**Proof:** Observe, that $x_i^i \in U(x_1^i, x_2^i)$ for $i \in \{1, 2\}$ and that

\[
\text{diam } U(x_1^i, x_2^i) \leq \max_{1 \leq i \leq 2} \text{diam } K(x_1^i, \tau - 1 \delta(x_1^i)) \leq \max_{1 \leq i \leq 2} 2 \cdot \delta(x_1^i) \leq \max_{1 \leq i \leq 2} \{ \delta(x_1^i) : 1 \leq i \leq 2 \}.
\]

Thus from the definition 1 we obtain the existence of numbers $r_i > 1/2$, $i \in \{1, 2\}$ such that:

\[(4) \quad m(U(x_1^i, x_2^i) \cap E(x_1^i)) = r_i \cdot m(U(x_1^i, x_2^i)), \text{ if } i \in \{1, 2\}. \]

If the intersection (3) were empty, then

\[(5) \quad m[U(x_1^1, x_2^1) \cap (E(x_1^i) \cup E(x_2^i))] = \sum_{i=1}^2 m[E(x_1^i) \cap U(x_1^i, x_2^i)] = \sum_{i=1}^2 m(U(x_1^i, x_2^i)) \cdot \frac{r_i}{2} > m(U(x_1^i, x_2^i))
\]

in spite of the fact that (3) is a measurable subset of $U(x_1^i, x_2^i)$. Consequently these three sets must have a point in common, say $x^3 \in E(x_1^i) \cap E(x_2^i) \cap U(x_1^i, x_2^i)$, which proves our
**Definition 3.** (cf. [2]). Let \( \delta : X \to \mathbb{R}_+ \) be a positive function and let \( K \) be a subset of \( X \). By a \( \delta \)-decomposition of \( K \) we shall mean a sequence of sets \( \{ K_n : n \in \mathbb{N} \} \), which is a relabelling of the countable collection:

\[
K^m_j := \{ x \in K : \delta(x) > 1/m \} \cap K \left( x^0_j, 2^{-1} m^{-1} \right),
\]

where \( \{ x^0_j, j \in \mathbb{N} \} \) is a countable dense set in \( X \).

The key features of such a decomposition are recapitulated in a subsequent lemma:

**Lemma 2.** Let \( \{ K_n : n \in \mathbb{N} \} \) be a \( \delta \)-decomposition of \( K \).

Then:

\[ (i) \bigcup_{n=1}^{\infty} K_n = K \]

\[ (ii) x^1, x^2 \subset K \implies d_x(x^1, x^2) < \min \{ \delta(x^i) : 1 \leq i \leq 2 \} \]

\[ (iii) \text{if } x^0 \text{ belongs to the closure of } K_n \text{ of } K \text{ then there are points } x \in K_n \text{ with } d_x(x^0, x) < 3^{-1} \min \{ \delta(x^0), \delta(x) \}. \]

**Proof:** If \( x \subset K \) then \( \delta(x) > m^{-1} \) for some positive integer \( m \) and \( d_x(x, x^0_j) < 2^{-1} m^{-1} \) for some \( j = j(x, m) \in \mathbb{N} \).

Thus \( x \in K^m_j =: K_n \) where \( n = n(m, j) = 2^{m-1} \cdot (2j - 1) \) and (i) is proved. If \( x^1 \in K_n := K^m_j \) then \( \delta(x^1) > m^{-1} \) whenever \( i \in \{1, 2\} \).

By the triangle inequality we have:

\[
d_x(x^1, x^2) \leq d_x(x^1, x^0_j) + d_x(x^0_j, x^2) < 2^{-1} m^{-1} + 2^{-1} m^{-1} = m^{-1} < \min \{ \delta(x^1) : i \in \{1, 2\} \} \text{ and (ii) is proved.}
\]
If $x_0 \notin \text{cl } K = \text{cl } \mathbb{R}^m$ then there is a sequence $x^k \in K = \mathbb{R}^m$ convergent to $x_0$. Let $r \in (0, 4^{-1} m^{-1})$ be a number such that $d(x_0^k, x_0^0) = 2^{-1} m^{-1} - 2r$ and let $d(x_k^k, x_0^0) < r$ for all $k > k_0$.

Thus $d(x_k^k, x_0^0) \leq d(x_k^k, x_0^0) + d(x_0^0, x_0^0) = 2^{-1} m^{-1} - 2r + r = 2^{-1} m^{-1} - r < 2^{-1} m^{-1}$ and consequently $x_k^k \in K(x_0^0, 2^{-1} m^{-1})$ for $k > k_0$. Moreover we have $\delta(x_k^k) > m^{-1}$ and for sufficiently large $k > k_0$, $d(x_k^k, x_0^0) < 4^{-1} m^{-1} < 3^{-1} \min \{\delta(x_k^k), \delta(x_0^0)\}$ since $x_k^k$ tends to $x_0$.

THEOREM 1. Each preponderantly equicontinuous family $F$ of functions $f : X \to Z$ has the property $A_2$.

Proof: Without loss of generality we can suppose that the functions from the family $F$ are uniformly bounded, i.e., there are a point $z \in Z$ and a positive number $M = M(z) > 0$ such that:

$$\{f(x) \in Z : (x, f) \in X \times F \} \subset K(z, M).$$

This follows from the fact that the formula (1) depends only on uniformity of the space $Z$ and thus the particular distance functions may be replaced by the uniformly equivalent ones, e.g., $d := \min \{d_Z, 1\}$. Assume by a way of contradiction that $F$ fails to have the $A_2$ property from definition 2 and yet is preponderantly equicontinuous in the meaning of definition 1.

Then there exists a closed set $K \subset X$ such that:

$$\bigwedge_{x_0 \in K} \bigvee_{\varepsilon(x) > 0} \bigwedge_{\delta > 0} \bigvee_{x \in K} [d(f(x), f(x_0)) < \varepsilon(x) \wedge d(x, x_0) < \delta].$$

In other words...
osc \ h(x) = \inf \left\{ \sup \left\{ D(h(x), h(x_0)) : x \in K(x_0, \delta) \right\} : \delta > 0 \right\} = 
\inf \left\{ \sup \left\{ d(f(x), f(x_0)) : (x, f) \in K(x_0, \delta) \times F \right\} : \delta > 0 \right\}.

We have $K_0 = \bigcup_{n=1}^{\infty} K_n$, where for $n=1, 2, \ldots$ the set $K_n$ is defined as follows:

$$K_n := \left\{ x \in K_0 : \text{osc} \ h(x) > n^{-1} \right\}.$$

The function $(\text{osc} \ h) : X \to R$ being upper semicontinuous, each of the sets $(11)$, $n \in N$, is closed in $X$. Since the set $K_0$ is complete, as a closed subspace of a complete metric space $X$, then by famous Baire Category Theorem one of the sets $K_n$, $n \in N$, by way of example $K_m$ is of the second category in $X$.

Let $A_m$ denotes the relative interior of $K_m$ in $K_0$ and take $Q := \text{cl} A_m$. We may assume that $Q$ is a nonempty perfect set contained in $X$, with the property that the oscillation $(10)$ of the restriction of $h$ to $Q$ exceeds $m^{-1}$ at every point of $Q$. Let $\delta_1$ be a positive function associated with multifunction $E$ in definition 1 and choose a further positive function $\delta_2 : X \to R_+$ so that:

$$D(h(x_1), h(x_2)) < 1/6m \text{ for any } x_2 \text{ belonging to } E(x_1) \text{ and satisfying } 0 < d_X(x_1, x_2) < \delta_2(x_1).$$

Let

$$\delta_3 := \min \left\{ \delta_1, \delta_2 \right\} \text{ and let } \left\{ Q^n : n \in N \right\} \text{ be a } \delta_3\text{-decomposition (see definition 3) of the set } Q. \text{ By Baire's category}$$
theorem invoked once again we can find that one of these sub-
sets, say $Q^k$ is dense somewhere:

$$\text{cl} [Q^k \cap V] \supset V$$

for certain subset $V$ relatively open in $Q$.

Let $x^3$, $x^4$ be any points in $Q \cap V$. By virtue of the density
of $Q^k$ and the item (iii) from Lemma 2 we may select the
points $x^3_k$, $x^4_k$ belonging to $Q^k$ so that:

$$d(x^1, x^4_k) < 3^{-1} \min \{ \delta_3(x^4), \delta_3(x^4_k) \} ; i \in \{3, 4\}.$$  \hspace{1cm} (13)

Define:

$$U(x, x^4_k) := K(x^1, 3^{-1} \delta_3(x^1)) \cap K(x^4_k, 3^{-1} \delta_3(x^4_k))$$  \hspace{1cm} (14)

for $i \in \{3, 4\}$ and observe that:

$$\text{diam } U(x^1, x^4_k) \leq 2/3 \min \{ \delta_3(x^1), \delta_3(x^4_k) \}. $$  \hspace{1cm} (15)

Then, by lemma 1, there are points $x^{1+2}_k \in E(x^1) \cap E(x^4_k) \cap

U(x^1, x^4_k)$. From the definition (14) of $U(x^1, x^4_k)$ we have

$$\max \{ d_X(x^{1+2}_k, x^1), d_X(x^{1+2}_k, x^4_k) \} <$$

$$< \min \{ \delta_3(x^1), \delta_3(x^4_k) \}. $$  \hspace{1cm} (16)

Consequently:

$$\max \{ D(h(x^{1+2}_k), h(x^1)), D(h(x^{1+2}_k), h(x^1)) \} \leq 1/6m.$$  \hspace{1cm} (17)

On the other hand, from the item (ii) of lemma 2, we have:

$$d_X(x^3_k, x^4_k) < \min \{ \delta_3(x^3_k), \delta_3(x^4_k) \}.$$  \hspace{1cm} (18)

Thus there exists a point $x^7_k \in E(x^3_k) \cap E(x^4_k) \cap U(x^3_k, x^4_k)$

where $U(x^3_k, x^4_k)$ is defined by a similar manner as in (14).

Therefore:

$$\max \{ d_X(x^7_k, x^4_k) ; j \in \{3, 4\} \} < \min \{ \delta_3(x^1_j) ; j \in \{3, 4\} \}.$$  \hspace{1cm} (19)

from which we obtain
(20) \( D(h(x_k^7), h(x_k^j)) < 1/6m \) for \( j \in \{3, 4\} \).

Combining (17) and (20) together we obtain by the triangle inequality:

(21) \[
D(h(x^3), h(x^4)) \leq D(h(x_k^3), h(x_k^4)) + D(h(x_k^3), h(x_k^4)) + \ldots + D(h(x_k^6), h(x^4)) < \ldots
\]

But this contradicts our choice of \( Q := cl A_m \) and \( m \in \mathbb{N} \), since (21) means that \( \text{osc } h(x) \) cannot be greater than \( 1/m \) for \( x \in Q \). Consequently (8) cannot be fulfilled and the family \( F \) must have the A-property, as required. Hence the proof of our theorem 1 is completed.

Collating theorems 0 and 1 together we obtain:

COROLLARY 1. Let \( g: X \times Y \to Z \) be a transformation whose all \( Y \)-sections \( \{g(\cdot, y): y \in Y\} \subset Z^X \) create a preponderantly equicontinuous family and all \( X \)-sections \( g_x := g(x, \cdot) \in \mathcal{L}Z^Y, x \in X \) belong to the Baire class \( \alpha, 0 < \alpha < \Omega \). Then \( g \) belongs to the Baire class \( \alpha \) too.

In my earlier paper [25] the transformations defined on the real line are investigated in a similar spirit. A notion of \( E \)-equicontinuity with respect to a system of path \( E: X \to 2^X \) satisfying the intersection condition (cf. [2]) is introduced and a result similar to the above corollary 1 is obtained in such framework. In particular approximative equicontinuity (cf. [7]) and \( I \)-approximative equicontinuity (i.e. related to the category analogue of the density topology introduced by
Wilczyński, see [28]) is covered. However note, that the uniformity generated by the density topology (see [21]) leads to the notion of approximative equicontinuity defined in [7] while the I-density topology of Wilczyński fails to be uniformizable. For the basis facts concerning uniform spaces the reader is referred to [23].

Taking into account that the property $A_2$ implies in turn the following property $A_3$ of the family $F \subseteq 2^X$:

\[(22) \quad F \in A_3 \iff \bigwedge_{x \in X} \bigwedge_{r > 0} \bigvee_{x_0 \in K(x,r)} [F \text{ is equicontinuous at } x_0] \]

and modifying in a suitable manner the theorem 5 from [13] we are able to obtain from our theorem 1 the following:

**COROLLARY 2.** Let $g : X \times Y \rightarrow Z$ be a transformation whose all $Y$-sections create a preponderantly equicontinuous family and all $X$-sections are densely continuous ($= cliquish$). Then $g$ is also densely continuous ($= cliquish$) as a transformation defined on the product space. Bearing in mind that we can always replace $d_z$ by an uniformly equivalent bounded distance function $d$ and slightly modifying the proof of theorem 7 from [13] we obtain immediately:

**THEOREM 2.** Any equi-upper semicontinuous family $F$ of functions $f : X \rightarrow \mathbb{R}$ has the property $A_2$. The same holds for equi-lower semicontinuity of $F$.

Let us recall (cf. [1],[4],[6],[9]) that a collection of functions $F \subseteq R^X$ is equi-upper semicontinuous at a point $x \in X$ if

\[(23) \quad \bigwedge_{\varepsilon > 0} \bigwedge_{\delta > 0} \bigvee_{f \in F} \bigwedge_{x \in X} [x \in K(x_0, \delta) \Rightarrow f(x) - f(x_0) < \varepsilon]. \]
The collection $F$ is equi-upper semicontinuous if (23) holds for every $x \in X$. Equi-lower semicontinuity is defined in a similar manner or by replacing $f$ by $-f$ in the formula (23).

At the present we are going to introduce a one-sided concept of preponderant equi-semicontinuity.

**DEFINITION** 4. A family $F_1$ of $m$-measurable real-valued functions $f: X \to \mathbb{R}$ is said to be preponderantly upper semiequicontinuous if there are a function $\delta: X \to \mathbb{R}_+$ and a multifunction $E$ exactly as in the definition 1 such that for all $x^0 \in X$ conditions (a) and (b) from definition 1 are both satisfied and moreover

$$(24) \quad \forall \varepsilon > 0 \quad \forall r > 0 \quad \forall f \in F_1 \quad \forall x \in X \quad x \in E(x^0) \cap K(x^0, r) \Rightarrow \quad f(x) \in (-\infty, f(x^0) + \varepsilon)]$$

Sometimes the values of $E$ are additionally demanded to be $F_0$ sets. A family $F_2 \subset \mathbb{R}^X$ is called preponderantly lower semi-equicontinuous if $F_1 := \{-f : f \in F_2\}$ is preponderantly upper semi-equicontinuous. If the above family $F_1$ include a single function $f_i$, $i \in \{1, 2\}$, then $f$ is called upper (resp. lower) preponderantly semicontinuous. Notice, that there are preponderantly non-continuous functions, but simultaneously both lower and upper preponderantly semicontinuous (see an example in [12]). Let us suppose at present that our space $Y$ is additionally endowed with a positive Borel measure $m_Y$ satisfying a condition analogous to the condition imposed on $m$.

The subsequent theorem is an analogue of the th. 8, p. 20 from [7]:
THEOREM 3. Let \( g: X \times Y \rightarrow \mathbb{R} \) be a function whose all 
\( Y \)-sections are approximately ([14],[20]) upper semicontinuous 
and \( F_2 := \{ g(x, \cdot) : x \in X \} \subset \mathbb{R}^Y \) is a preponderantly upper semi-
equicontinuous family. Then \( g \) is preponderantly upper semi-
continuous on the product space \( X \times Y \) endowed with the tensor 
product \( m \otimes m_Y \) of measures.

Proof: Let \((x^0, y^0) \in X \times Y \) and let \( \varepsilon > 0 \) be given. There 
is a number \( r_1 > 0 \) such that

\[
(25) \quad g(x, y^0) \in (-\infty, g(x^0, y^0) + \varepsilon/2) \quad \text{whenever} \quad x \in E(x^0) \cap K(x^0, r_1) \quad \text{and} \quad m(E(x^0) \cap U(x^0)) > (1-t) m(U(x^0)) \text{ if} \\
\text{diam } U(x^0) < \delta^2(x^0, t).
\]

On the other hand, by the preponderant upper semiequicontiunity 
of the family \( F_2 \) we have:

\[
(26) \quad g(x, y) \in (-\infty, g(x,y^0) + \varepsilon/2) \quad \text{whenever} \quad x \in E(x^0) \cap K(x^0, r_1) \quad \text{and} \quad y \in E(y^0) \cap K(y^0, r_2) \quad \text{for a suitable, suffi-
ciently small} \quad r_2 > 0.
\]

Define \( E^2(x^0, y^0) := E(x^0) \times E^1(y^0) \). For all \((x,y)\) belonging 
to the intersection \( E^2(x^0, y^0) \cap K((x^0, y^0), r_3) \) where

\[
r_3 := \min \{ r_1 : i \in \{1, 2\} \}
\]

we have

\[
g(x,y) - g(x^0, y^0) = g(x,y) - g(x,y^0) + g(x,y^0) - g(x^0, y^0) < \varepsilon/2 + \varepsilon/2 = \varepsilon,
\]

so that \( g(x,y) \in (-\infty, g(x^0, y^0) + \varepsilon) \).

If \( U^2(x^0, y^0) \) is contained in \( U(x^0) \times V(y^0) \) then:

\[
(27) \quad m_2 [U^2(x^0, y^0) \cap E^2(x^0, y^0)] \\
\geq \frac{\varepsilon}{2} m_2 [U(x^0) \times V(y^0)] \cap (E(x^0) \times E^1(y^0)] = \\
\geq m_2 [U(x^0) \cap E(x^0) \times V(y^0) \cap E^1(y^0)] = m[U(x^0) \cap \\
\cap E(x^0)] m_Y [V(y^0) \cap E^1(y^0)] = (1-t) m(U(x^0)) \cdot \\
(1/2 + t) m (V(y^0)) = [1/2 + t(1/2-t)] m_2(U(x^0) \times V(y^0)) > 
\]
whenever $\text{diam } V(y^0) < \delta^1(y^0)$. The sign $m_2$ means here $m \otimes m_Y$ and in $X \times Y$ the distance function $d_3((x^1, y^1), (x^2, y^2)) = \max \{d_X(x^1, x^2), d_Y(y^1, y^2)\}$ is selected.

Obviously a theorem similar to theorem 3 holds for functions with preponderantly lower semiequicontinuous sections (cf. [7], th. 9). The next theorem is in spirit of famous Kempisty's result [16]. We need the following lemma:

**Lemma 3.** Suppose that a function $g : X \times Y \rightarrow \mathbb{R}$ has all of its $Y$-sections preponderantly lower semicontinuous (not necessarily equisemicontinuous!). Then for each positive real constant $s$ the function $g_s : X \times Y \rightarrow \mathbb{R}$ defined by the formula:

$$g_s(x^0, y^0) := \sup \{g(x^0, y) : y \in K(y^0, s)\}$$

is preponderantly lower semicontinuous on the product space $(X \times Y, d_3, m_2)$ where $m_2 := m \otimes m_Y$ and $d_3$ is defined at the end of the proof of theorem 3.

**Proof:** Let $(x^0, y^0) \in X \times Y$ be an arbitrary fixed point and let $\varepsilon > 0$ be given. By (28) there is a point $y^1$ belonging to the ball $K(y^0, s)$ such that $g(x^0, y^1) \in (g_s(x^0, y^0) - \varepsilon, +\infty)$. Since the section $g(\cdot, y^1)$ is preponderantly lower semicontinuous on $X$, there exists a radius $r_1 > 0$ such that for each $x \in E(x^0) \cap K(x^0, r_1)$ we have $g(x, y^1) \in (g_s(x^0, y^0) - \varepsilon, +\infty)$. Since $d_Y(y^0, y^1) < s$, there exists a number $r_2 > 0$ such that $d_Y(y^0, y^1) = s - r_2$. By the triangle inequality we have:

$$d_Y(y^1, y) \leq d_Y(y^1, y^0) + d(y^0, y) < (s - r_2) + r_2 = s$$
for each \( y \in K(y^0, r_2) \). Thus \( y^1 \) belongs to the ball \( K(y, s) \) whenever \( y \in K(y^0, r_2) \). Consequently \( g(x, y^1) \leq g_s(x, y) \) whenever \( y \in K(y^0, r_2) \) and \( x \in E(x^0) \cap K(x^0, r_1) \). But \( g(x, y^1) \in \mathcal{E}(g_s(x^0, y^0) - \varepsilon, +\infty) \) so also \( g_s(x, y) \in (g_s(x^0, y^0) - \varepsilon, +\infty) \) for all \( (x, y) \in E(x^0) \cap K(x^0, r_1) \) and \( K(y^0, r_2) \cap E^2(x^0, y^0) \cap K((x^0, y^0), r_3) \) where \( E^2(x^0, y^0) := E(x^0) \times K(y^0, r_2) \), \( r_3 := \min \{ r_1, r_2 \} \in [1, 2] \) and \( K((x^0, y^0), r_3) = K(x^0, r_3) \times K(y^0, r_3) \) gives a specific choice of a distance function \( d_2 \) on \( X \times Y \). Observe that \( (x^0, y^0) \in E^2(x^0, y^0) \cap E(x^0, y^0) \) and that \( m_2(U(x^0) \times V(y^0) \cap E(x^0, y^0)) = m(U(x^0) \times E(x^0, y^0)) \).

\[
\begin{align*}
m_Y(V(y^0) \cap K(y^0, r_2)) > 2^{-1} m(U(x^0)) \quad \text{and} \quad m_Y(V(y^0)) = 2^{-1} m_2(U(x^0) \times V(y^0))
\end{align*}
\]

whenever \( V(y^0) \subseteq K(y^0, r_2) \). Hence \( m_2(U^2(x^0, y^0) \cap E^2(x^0, y^0)) > 2^{-1} m_2(U^2(x^0, y^0)) \) provided \( \text{diam } U^2(x^0, y^0) < \delta^2(x^0, y^0) := \min \{ \delta(x^0), r_2 \} \) where \( \delta \) is a function from the item (b) of def. 1. Since \( (x^0, y^0) \in X \times Y \) was arbitrary, we have defined a multifunction \( (x^0, y^0) \mapsto E^2(x^0, y^0) \) and two positive functions \( (x^0, y^0) \mapsto \delta(x^0, y^0), (x^0, y^0) \mapsto r(x^0, y^0) := r_3 \) satisfying mutatis mutandis all requirements of definition 4. Observe however that \( E^2(x^0, y^0) \in F_5(X \times Y) \) iff \( E(x) \in \mathcal{E}F_5(X) \). Thus \( g_s : X \times Y \rightarrow \mathbb{R} \) is preponderantly lower semicontinuous jointly, as a function of two variables.

**THEOREM 4.** Let \( g : X \times Y \rightarrow \mathbb{R} \) be a function whose all \( Y \)-sections are preponderantly lower semicontinuous and all \( Y \)-sections are \( d_Y \)-upper semicontinuous. Then \( g \) is a limit of a decreasing sequence of preponderantly lower semicontinuous functions.

**Proof:** Take an arbitrary sequence \( s_1 > s_2 > \ldots > 0 \) tending decreasingly to zero and observe that because of the assu-
med \( d_Y \) - upper semicontinuity of \( Y \)-sections we have
\[
g(x, y) = \lim_{s \to 0^+} g_s(x, y) = \lim_{n \to \infty} g_n^s(x, y)\]
where \( g_n \) are defined by (28). Moreover for all \( n \in \mathbb{N} \) the following inequality:
\[
(30) \quad \sup_n g_s^* \mathcal{K}(y, s_{n+1}) \leq g_{n+1}^s(x, y) = \sup_n g_n^* \mathcal{K}(y, s_n)
\]
holds, since \( \mathcal{K}(y, s_{n+1}) \subseteq \mathcal{K}(y, s_n) \) for \( y \in Y \). That observation achieves the proof. Under the continuum hypothesis one can construct a nonmeasurable function \( g : X \times Y \to \mathbb{R} \) with approximately lower semicontinuous \( X \)-sections and approximately upper semicontinuous \( Y \)-sections. Let us remark that paper [11] contains a theorem similar to our theorem but concerning qualitative semicontinuity under the following rather artificial condition imposed upon \( d_Y \):
\[
(31) \quad \forall y \in K(y_0, \text{dist}(y_1, \text{Fr} K(y_0, r))) \Rightarrow y_1 \in K(y, r).
\]
An inspection of our proof shows that the condition (31) in [11] is superfluous. Our method also allows us to generalize onto the case of arbitrary metric spaces the theorem 6 from [7] in which the space \( Y \) is needlessly assumed to be euclidean and finite-dimensional. Finally, we give a theorem related to the results from [3] and [24].

**Definition 5 (cf. [27]).** A transformation \( f : X \to Z \) is said to be non-alternating (in the sense of Whyburn) if, whenever \( C \) is connected in \( Z \), its inverse image \( f^{-1}(C) \) is connected in \( X \).

Observe, that in the case where \( X=Z=\mathbb{R} \) Definition 5 reduces
to \( f \) being (weakly) increasing or decreasing.

In the sequel we shall assume additionally that the space \( Z \)
has in addition the property, that each ball in \( Z \) is connected,
and that \( X = \mathbb{R} \).

**Theorem 5.** Let \( f : X \times Y \rightarrow Z \) be a transformation whose
all \( Y \)-sections are non-alternating and all \( X \)-sections create
a separable subspace of the space \( B_1(Y, Z) \) of Baire 1 bounded
transformations. Then \( f \) is also of the first Baire class.

**Proof:** Let us put \( h(x) := f_{x} \in B_1(Y, Z) \). We prove that \( h \)
is a transformation of the first Baire class. Since the target
space \( h \cdot X \) is separable, each open set in this image is a
countable union of open balls. On the other hand each open ball
\( K(g, r) \) is a countable union of the closed balls \( \overline{K}(g, r-2^{-n}) \),
\( n=1,2,... \). Therefore it suffices to prove that inverse images
\( h^{-1}(\overline{K}(g, r-2^{-n})) \) are subsets of \( X \) of the type \( F_\sigma \).

Indeed, we have:

\[
(32) \quad h^{-1}(\overline{K}(g, s)) = \{ x \in X \mid d(h(x), g) \leq s \} = \{ x \in X \mid d_1(f(x,y), g(y)) \leq s \text{ for each } y \in Y \} = \bigwedge_{y \in Y} (f^{-1}(Y))^{-1}(\{ z \in Z \mid d_1(z, g(y)) \leq s \}).
\]

All the balls \( \overline{K}(g(y), s) \subset Z \) are connected on the strength of
our additional assumption imposed upon the space \( Z \). Bearing in
mind, that the section \( f^Y \), \( y \in Y \) are non-alternating, we con-
clude without difficulty that \( (f^Y)^{-1}(K(g(y), s)) \) is connec-
ted and thus also convex, provided that \( X \) is the real line.

Hence \( h^{-1}(\overline{K}(g,s)) \) is convex as the intersection of the in-
dexed family of convex sets. Since each convex subset of the
real line is ambiguous, therefore \( h^{-1}(U) \in F_\sigma(X) \) for each open
subset \( U \subset h \cdot X \) provided \( U \) is a countable union of closed
balls. Consequently \( h : X \to B_1(Y, Z) \) is of the first Baire class and has the separable range. Observe that \( f(x, y) = h(x)(y) \) so that, by virtue of Baire theorem, the \( Y \)-sections of \( f \) fulfil the property \( A_2 \). Invoking the theorem 5 with \( \alpha = 1 \) we obtain the claimed assertion. Note, that the space \( X \) may be generalized to be e.g. a curve in euclidean space, in particular a circle, i.e. a topological space without no order relation compatible with topology.

**COROLLARY 3.** Assume additionally that \( Y \) is compact metric space. Let \( f : X \times Y \to Z \) be a transformation with non-alternating \( Y \)-sections and continuous \( X \)-section. Then \( f \) is in the first Baire class.

**Proof:** The space \( C(Y, Z) \) endowed with the uniform metric

\[
D(\varepsilon_1, \varepsilon_2) := \sup \{d_1(\varepsilon_1(y), \varepsilon_2(y)) : y \in Y\}
\]

is separable in the presence of compactness of \( Y \) and separability of \( Z \). Thus we may apply the last theorem 5. In case where \( Z = R = \bigcup_{k=-\infty}^{+\infty} [-k, k] \) this corollary gives a negative answer to the question 3 \( a, g \) from [10]. In connection with Corollary 3 let us recollect, that by an old result of H.D. Ursell [26] a function \( f : R^2 \to R \) with isotonic \( Y \)-sections and \( L \)-measurable \( X \)-sections is \( L \)-measurable on the plane. Obviously this result may be generalized in a style of theorem 5. On the other hand a function \( f : R^2 \to R \) with nondecreasing both \( X \)-sections and \( Y \)-sections may fails to be Borel measurable.

Paper [24] contains an example of function defined on the plane not belonging to the first Baire class, whose all \( X \)-sections are right-continuous and increasing while all \( Y \)-sections are
decreasing.
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REFERENCES

[7] Grande Z., La mesurabilité des fonctions de deux variables et de la superposition F(x, f(x)), Dissertationes Math. CLIX (1978), 1-50


[22] Saks S., Theory of the integral, Monografie Mat. 7, PWN,
O PRZEWYŻŞAJĄCO JEDNAKOWO CIĄGŁYCH RODZINACH PRZEKSZTALTCAŃ

Streszczenie

W pracy tej pokazano, że przewyższaçco jednakowo ciągła rodzina przekształceń mierzalnej przestrzeni metrycznej w ośrodkową przestrzeń metryczną posiada wprowadzoną przez Grandego własność $A_2$. Jako wniosek otrzymuje się pełne rozwiązanie problemu 11 opublikowanego w trzecim zeszytcie Problemmów Matematycznych [10]. Wprowadzono również pojęcie przewyższająco jednakowo półciągłej rodziny odwzorowań i udowodniono 2 proste fakty dotyczące tego pojęcia. Pracę kończy twierdzenie o przynależności do pierwszej klasy Baire'a pewnego odwzorowania określonego na przestrzeni produktowej i o wartościach w przestrzeni metrycznej, stanowiące uogólnienie wcześniejszego wyniku autora [24].