ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ w BYDGOSZCZY
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ON FRAGMENTS OF THE METRIC GEOMETRY ON THE SPHERE

In my previous paper [5] some system of axioms of the metric geometry on the sphere was presented. One should not confuse the metric geometry on the sphere with the elliptical geometry.

It was proved, that such system of axioms has a fixed model, called a fundamental model $\mathcal{S}_o$. In that manner the proof of theorem I A from [5] has been completed. To prove that each model of our system of axioms is isomorphic with $\mathcal{S}_o$, a theory built on the axioms A1 - A3 must be developed. In the present paper we want to realize such a program, while the categoricalness of our system of axioms will be the subject of a later paper. We repeat our axioms for the later use:

A1 $[a \geq 0 \wedge b \geq 0 \wedge (|a-b| \leq AB \leq a+b) \wedge (a+b + AB \leq 2\pi)] \Longleftrightarrow$
$\Longleftrightarrow \bigvee C \ (CA = a \wedge BC = b)$,

A2 $[a > 0 \wedge b > 0 \wedge (|a-b| < AB < a+b) \wedge a+b + AB < 2\pi] \Rightarrow$
$\Rightarrow \bigvee 2 \ C_i \ (AC_i = a \wedge BC_i = b, \ i=1,2)$,

A3 $[A \neq B \wedge AB + BC = AC \wedge \cos AB \cdot \cos BD = \cos AD] \Longrightarrow$
$\Longrightarrow \cos CB \cdot \cos BD = \cos CD$

In those axioms the capital letters denote allways the points of the sphere, while the small letters denote the real numbers, and the sign AB denotes the distance between points A and B

In this paper usual logical symbols will be applied. The symbol like $\bigvee_{nX}$ means: there is exactly n points X such that ....

Basing on these axioms A1–A3 one can define all notations of the metric geometry on the sphere and develop easily all this geometry. In the present paper we confine ourselves to develop it only to such a degree as is indispensable to prove the completness of the above system of axioms.

We shall use freely the results, notations and formulae from our previous paper [5] .

Directly from the axiom A1 result the following three theorems:

T1 $AB \leq BC + CA$

T2 $AB + BC + CA \leq 2\pi$

T3 $AB \geq 0$

T4 $AA = 0$

PROOF. From theorems T3 and T2 results the inequality $0 \leq AA \leq (2/3)\pi$. On the other hand, if we assume that $AA = a > 0$ and $b = a/2$ then by virtue of A2 there exists a point C such that $AC = a$ and simultaneously $AC = b$, hence $AC = a/2 = a$ which proves that $AA = a = 0$. The axiom A0 from the paper [5], guaranteeing the existence of two distinct points, whose distance differs from zero and at the same time differs from π , is dependent on the remaining ones. Indeed, from A1 and T4 we infer immediately:

T0 $\bigvee_{A,B} \quad 0 \neq AB \neq \pi$

Next, we have :

T5 $AB = BA$.

PROOF. From T1 we deduce the inequalities $AB \leq BA + AA$ and $BA \leq AA + AB$. Those inequalities are equivalont to $AB \leq BA$ and $BA \leq AB$ respeotively. Thus the claimed equality $AB = BA$ occurs.

In the sequel we shall use frequently the theorem 5 without invoking its number. Subsequently, we have:

T6 $AB \leq \pi$

PROOF. From T1 we infer the inequality $AB \leq BA + AA$ and from T2 the inequality $AB + BA + AA \leq 2\pi$. Combining both inequalities we obtain the claimed assertion, that $AB \leq \pi$.

Collating the theorems T3 and T6 we get directly:

T7 $0 \leq AB \leq \pi$.

T8 $AB = 0 \Longrightarrow A = B$

PROOF. Assume that $AB = 0$ and $A \neq B$. From A1 results the existence of 2 points C and D such that $AC = BC = BD = \pi/6$, but the mutual distance between those points equals $CD = \pi/3$. Since

$$AD \leq DB + BA \quad \text{and} \quad DB \leq BA + AD \quad \text{then} \quad AD = \pi/6 .$$

By the above considerations it follows, that all requirements of our axiom A3 are fulfilled, viz.

$A \neq B$, $AB + BC = AC$ and $\cos AB \cos BD = \cos AD$, but simultaneously we have $\cos CB \cos BD \neq \cos AD$, in a marked contradiction with A3. Consequently points A i B cannot be distinct.

T9 $\bigwedge_{A} \quad \bigvee_{1\,A^*} \quad AA^* = \pi$

PROOF. The existence of such a point A^* is assured by A1 and

T4. Assume, on the contrary, that there exist two distinct points A^* and A_1^* such that $AA^* = AA_1^* = \pi$. Then theorems T8 and T3 yield the inequalities $A^*A_1^* > 0$ and $AA^* + A^*A_1^* + A_1^*A \geqslant 2\pi$; that leads to a contradiction with T2, so that the point $A^* = A_1^*$ is unique and the proof of T9 is complete. The validity of T9 insures the corectness of the following definition:

Df 1 $\quad \alpha(A) = A^* \overset{df}{\Longleftrightarrow} AA^* = \pi$.

The points A and A^* from Df 1 are called antipodal. In the sequel a capital letter with the asterisk as an upper index will be denote allaways a point antipodal with respect to a point named by the same letter but without the asterisk. Directly from the established definition Df 1 results:

T 10 $\quad \alpha(A) = A^* \Longleftrightarrow \alpha(A^*) = A$

T 11 $\quad \alpha(A) = A^* \Longrightarrow \bigwedge_X \; AX + XA^* = \pi$

PROOF. As an immediate corollary from Df 1 and T1 we obtain the inequality $\pi \leq A^*X + XA$. Simultaneously by virtue of Df 1 and T2 we have the oposite inequality $\pi \geqslant A^*X + XA$. From here our assertion easily follows.

Df 2. $w(A, B, C) \overset{df}{\Longleftrightarrow} [AB = AC + CB \vee BC = BA + AC \vee AC = AB + BC \vee AB + BC + CA = 2\pi]$.

The relation $w(A,B,C)$ is called collinearity of points A, B and C on the sphere. Directly from the above definition Df 2 result

T 12 $\quad w(A, B, C) \Longleftrightarrow w(C, A, B) \Longleftrightarrow w(B, C, A)$

T 13 $\quad w(A, A, B)$

In the futurity as in the case of T5, the theorems T 12 and

T 13 will be invoked in our considerations without any mentioning of its tabels. As a corollary from the lemma L 12 (see [5]) and from T7 we get the following:

T 14 $w(A,B,C) \Longleftrightarrow Q(AB, AC, BC) = 0$

Again, we have:

T 15 $w(A_1,A_2,A_3) \Longrightarrow \bigvee_B [A_iB = \pi/2 \quad (i=1,2,3)]$

PROOF. We must consider only the case, when $A_1 \neq A_2 \wedge A_1 \neq A_3 \wedge A_2 \neq A_3$. One can omit the trivial cases, in which our assertion follows immediately from A1. With regard to the Df 2 it becomes to take under considerations four possibilities. Let us inquire in particular one of those possibilities, namely $A_1A_2 = A_1A_3 + A_3A_2$. Then by virtue of A1 and T7 there is a point B such that $BA_1 = A_3B = \pi/2$. In turn, the assumption, that $A_2B \neq \pi/2$ leads to a contradiction with the axiom A3.

Analogously we treat the cases, where

$A_2A_3 = A_2A_1 + A_1A_3$ or $A_1A_3 = A_1A_2 + A_2A_3$.

If $A_1A_2 + A_2A_3 + A_3A_2 = 2\pi$, then bearing in mind T 11, we infer the equality $A_1A_2 = A_1A_3^* + A_3A_2$. Finally, from the above there exists a point B such that $A_1B = A_2B = A_3^*B = \pi/2$. Since we have $A_3^*B = \pi/2$, then by virtue of T11 is $A_3B = \pi/2$.

Due to T 14, Df 1 and L 2 (from [5]) one can easily prove the subsequent 2 theorems:

T 16 $w(A,B,C) \Longrightarrow w(A^*, B, C)$,

T 17 $\bigwedge_X w(A, A^*, X)$

As a direct consequence of T 11 and T 10 we obtain the equivallency

T 18 $\quad AB + BC = AC \Leftrightarrow BC + CA^{*} = BA^{*}$

T 19 $\quad [A_i B = \pi/2, \quad i = 1,2,3] \Longrightarrow w\ (A_1, A_2, A_3)$

PROOF. In the case where the points A_i $(i=1,2,3)$ are not all distincts, the thesis follows from T 13.

If the points $A_i (i=1,2,3)$ are pairwise distinct and if we additionaly assume the following inequalities $A_1A_2 \leq A_2A_3$, $A_1A_2 + A_2A_3 \leq \pi$ then applying an A1, we can find two points C_1 and C_2 whose distance from the points A_1 and A_2 equals respectively : $A_1C_1 = A_1A_2 + A_2A_3$, $A_2C_1 = A_2A_3$, $A_1C_2 = A_2A_3 - A_1A_2$, $A_2C_2 = A_2A_3$. The latest four equalities yields:

(1) $(A_1A_2 + A_2C_1 = A_1C_1 \wedge A_2A_1 + A_1C_2 = A_2C_2)$.

Supporting on A3, (1) and on the assumption of our T 19 we deduce the equalities $BC_i = \pi/2$, $i = 1,2$, The distance between the points C_1, C_2 and A_3 and points A_2, B equals resp. A_2A_3 and $\pi/2$. In turn, A2 implies the existence of exactly two such points and, by this reason

(2) $\quad C_1 = A_3 \vee C_2 = A_3$.

From (1), (2) and Df 2 it follows the claimed assertion. In the oposite case, when $A_1A_2 + A_2A_3 > \pi$, applying the theorem T11 we get $A_1^{*}A_2 + A_2A_3^{*} < \pi$ and then the collinearity of points A_1, A_2 and A_3 holds. It suffice to invoke the T 16 for stipulation the collinearity of points A_1, A_2 and A_3. This complete the proof.

T 20 $\quad [0 < AB < \pi \wedge w\ (A,B,C) \wedge w(A,B,D)] \Rightarrow [w(A,C,D) \wedge w(B,C,D)]$

PROOF. From the assumption $w(A,B,C)$ we infer by T 15 the existence of certain point E such that $AE = BE = CE = \pi/2$. In the presence of T 9 and T 11 the equalities $AE^* = BE^* = CE^* = \pi/2$ also hold. In virtue of T 15 and the assumption $w(A,B,D)$ there exists a point F such that $AF = BF = DF = \pi/2$. In that manner we obtain three points E, E^* and F whose distance from A and B equals $\pi/2$. On the other hand the axiom A2 permit to construct exactly two such points. Thus the only possibility is either $F = E$ or $F = E^*$. From the above considerations by T 19 the thesis of T 20 follows.

$$T21 \; [0 < AB < \pi \;\wedge\; a,b \in [0,\pi] \wedge 0\,(a,b,AB) = 0] \implies$$
$$\implies \bigvee_{1C} (AB = a \wedge BC = b)$$

PROOF. Under the above assumption, the existence of such a point C is a consequence of a lemma L 13 and an axiom A1. Suppose, that there exist two distinct points C_1 and C_2 such that $AC_i = a$ and $BC_i = b$. Then by virtue of T 14, T 20 and Df 2 the following relations holds:

$$w(A,B,C_1),\; w(A,B,C_2),\; w(C_1,C_2,A) \text{ and } w(C_1,C_2,B).$$

Therefore, by L2 and L 13 a system of equations :

$$1 + 2\cos a \cos b \cos AB - \cos^2 a - \cos^2 b - \cos^2 AB = 0$$
$$1 + 2\cos^2 a \cos C_1C_2 - 2\cos^2 a - \cos^2 C_1C_2 = 0$$
$$1 + 2\cos^2 b \cos C_1C_2 - 2\cos^2 b - \cos^2 C_1C_2 = 0$$

must be satisfied. For $a \neq b$ and $a+b \neq \pi$ the above system has no solutions, so that the assumptions that $C_1 \neq C_2$ is not true. Also a couple of solutions $(a=b;\; AB=0)$ and $(a+b = \pi;\; AB = \pi)$ is not conformable with the assumption of T 21, since $0 < AB < \pi$.

Now, we shall prove the uniqueness of the existence of a point C in case when $a = b$ and $AB = a+b$. Profiting from the first part of our proof, we are able to construct in an unique way two points D_1 and D_2 in such a manner, that $AD_i = (2/3)a$ and $C_iD_i = (1/3)a$, $i = 1,2$. By T1 we infer the inequalities $AD_i + D_iB \geqslant AB$ and $D_iC_i + C_iB \geqslant D_iB$, from here under the assumptions $AB = 2a$, $C_iB = a$ it follows, that the equality $D_iB = (4/3)a$ occurs.

Since $AD_i + BD_i = AB$ and $AD_i = BD_i$ then by the preceding part of our theorem, we have $D_1 = D_2 = D$. Analogously $C_1 = C_2$. provided $DC_i + BC_i = BD$ and $DC_i \neq BC_i$.

A contradiction with the additional assumption yields the desired uniqueness of our point C.

The proof of uniqueness of C in the three remaining solutions $(a=b\ ;\ a+b+AB = 2\pi)$, $(a+b = \pi\ ;\ a-b = AB)$ and $(a+b = \pi;\ b-a = AB)$ is quite analogous and thus will be ommited. Indeed, by T11 and T9 those cases reduce to the already investigated ones.

T22 $(0 < AB < \pi \wedge 0 < c < \pi) \Rightarrow \bigvee_{2C_i} [AC_i = c \wedge w(A,B,C_i),(i=1,2)]$

PROOF. An equation $Q(x,c\ AB) = 0$ under the constraint c, $AB \in (0,\pi)$ possess two distinct solutions x_1 and x_2. In that case, in compliance with T21 there exist exactly two distinot points C_1 and C_2 such that $AC_i = c$ and $BC_i = x_i$. Since the distances between the points A, B and C fulfil the condition $Q(BC_i, AC_i, AB) = 0$, therefore by T14 these points are collinear, i.e. $w(A,B,C_i)$ holds.

Besides defining w we define one more ternary relation:

Df 3 $\perp ABC \overset{df}{\Longleftrightarrow} \cos AB \cos BC = \cos AC$.

The relation $\perp ABC$ is called perpendicularity of points

A,B,C. Directly from the definition Df3 result:

T 23 $\perp AAB \wedge \perp ABB$

T 24 $\perp ABC \Longleftrightarrow \perp CBA$

T 25 $(\perp ABC \wedge \perp ACB) \Longrightarrow (BC = 0 \vee BC = \pi \vee AB = AC = \pi/2)$

Theorems T23-T25 will be used in the sequel without any refering to its labels.

T 26 $[(1)\ 0 < AB < \pi \wedge (2)\ w(A,B,C) \wedge (3) \perp ABD] \Longrightarrow \perp CBD$

PROOF. From (1), T3, T4 and T8 it follows, that (4) $A \neq B$ and by (3) and Df 3 we obtain (5) $\cos AB \cos BD = \cos AD$. From (2) results by Df 2 and (2) that

(6) $AC = AB + BC \vee$ (7) $AB = AC + CB \vee$ (8) $BC = BA + AC \vee$ (9) $AB + BC + CA = 2\pi$. In the first case, namely when (6) occurs, taking under considerations formulae (4) and (5) we obtain the thesis from A3 and Df 3. The remaining cases may be treated in an analogous manner, hence we restrict ourselves to the oase when (7) occurs. From (7) and (5) by T11 we infer (10) $A^*B + BC = A^*C$ and (11) $\cos A^*B \cos BD = \cos A^*D$. Since (1) holds then, bearing in mind T11 the inequality $0 < A^*B < \pi$ holds. In view of T3, T4 and T8 we infer from the above inequality that (12) $A \neq B$. Formulae (12), (10), (11) and A3 attract $\cos CB \cos CD = \cos BD$ so, in view of Df 3 is equivalent to our assertion.

T 27 $[(1)\ 0 < AB < \pi \wedge (2) \perp ABC_i\ (i=1,2)] \Longrightarrow w(B,C_1,C_2)$

PROOF. If the point B coincidence with point C_1 and C_2 or i one of those points C_1, C_2 is antipodal with respect to B, then the thesis of our T 27 follows directly from T 13 or, resp. from T 17. Let us suppose that (3) $0 < BC_i < \pi$ and denote

(4) $BC_i = c_i$. From (3),(4) and T 22 it follows the existence of exactly two distinct points D_1 and D_2 such that (5) $BD_i = c_2$ and (6) $w(B, C_1, D_i)$ is verified. Applying (3),(6), (2), T 26 and Df 3 we obtain: (7) $\cos D_iB \cos AB = \cos D_iA$. From (2) and Df 3 we infer that: (8) $\cos AB \cos BC_2 = \cos AC_2$. Next, from the conditions (4),(5),(7) and (8) by T7 it follows that (9) $AD_i = AC_2$. Denoting $d = AC_2$ we infer from (1),(3) and (8) that (10) $0 < d < \pi$. By virtue of (1),(3), (4), (10), (8) and L2 we deduce that $Q(c_2, d, AB) = 4^{-1} \sin^2 AB \sin^2 BC_2 > 0$. Using L 12, it is not hard to see, that the above inequality is equivalent to (11) $[(|d-c_2| < AB < d+c_2) \wedge d + c_2 + AB < 2\pi]$. As a result of (3), (4) and (11) all assumptions of A2 are fulfilled. Therefore there exist exactly two points whose distance from A and B equals respectively d and c_2. From here taking into account (9), (5) and (4) two possibilities may happen: $C_2 = D_1$ or $C_2 = D_2$. Hence by (6) we obtain the desired thesis. To facilitate our computations, we shall work with the following functions η :

Df 4. If $0 < AB < \pi$, then put

$$\eta_{AB}(C) \overset{df}{=} (\cos BC - \cos AB \cos AC)\,(\cos^2 AC + \cos^2 BC - 2 \cos AB \cdot \cos AC \cdot \cos BC)^{-1/2} .$$

Moreover $\eta_{AB}(C) = 0$ for $AC = BC = \pi / 2$.

From Df 4 result directly the two next theorems:

T 28 $\bigwedge_{ABC} [0 < AB < \pi \implies \bigvee_{1 x} \; x = \eta_{AB}(C)]$

T 29 $0 < AB < \pi \Rightarrow \eta_{AB}(C) = \eta_{A^*B}(C) = - \eta_{AB^*}(C) = - \eta_{AB}(C^*)$

T 30 $0 < AB < \pi \Rightarrow |\eta_{AB}(C)| \leq \sin AC$

PROOF. From T1, T2 and T7 we obtain respectively:

$|AC - BC| \leq AB \leq AC + BC$, $AB + BC + CA \leq 2\pi$ and $AC, BC \in [0,\pi]$.

It is not hard to see, that the assumptions of lemma L 11 are satisfied and that $Q(AB, AC, BC) \geq 0$. The last inequality implies, by L2, that $1+2 \cos AB \cos BC \cos AC - \cos^2 AB - \cos^2 BC - \cos^2 AC \geq 0$, from there after some easy algebraic transformations we can obtain the promised thesis. Now, we can establish:

Df 5 $\quad 0 < AB < \pi \implies [\sin \xi_{AB}(C) = \eta_{AB}(C) \wedge \operatorname{sgn} \cos \xi_{AB}(C) =$

$\quad\quad = \operatorname{sgn} \cos AC \wedge -\pi < \xi_{AB}(C) \leq \pi]$.

The number $\xi_{AB}(C)$ is called the coordinate of the point C relative to points A and B. Directly from this definition Df 5 result:

T 31 $\quad 0 < AB < \pi \implies \xi_{AB}(C) = \begin{cases} \arcsin \eta_{AB}(C) & \text{if } AC \leq \pi/2 \\ k\pi - \arcsin \eta_{AB}(C) & \text{if } AC > \pi/2, \end{cases}$

where $k = \begin{cases} 1 & \text{if } \eta_{AB}(C) \geq 0 \\ -1 & \text{else.} \end{cases}$

T 32 $\bigwedge_{ABC} [0 < AB < \pi \implies \bigvee_{1x}\ x = \xi_{AB}(C)$.

As a corollary from theorems T 29, T 11, T 31, and Df 5 we obtain the subsequent theorem:

T 33 $\quad 0 < AB < \pi \implies \xi_{AB}(C^*) = \xi_{AB}(C) - k\pi$,

where k attains the same values as in T 31.

T 34 $\;(1)\; 0 < AB < \pi \implies (2)\; x = \xi_{AB}(C) \wedge (3)\; w(A,B,C)] \iff$

$\quad\quad \iff [(4)\; AC = |x| \wedge (5)\; \cos BC = \cos (AB - x)]\}$

PROOF. From (3), T 4 and L 13 results the equality (6)

$1 + 2 \cos AB \cos BC \cos AC - \cos^2 AB - \cos^2 AC - \cos^2 BC = 0.$

The left side of (6) may be described in the shape of:

$[\cos BC - \cos (AB + AC)][\cos BC - \cos (AB - AC)] = 0$.

Thus there exists a number e such that (7) $e^2 = 1$ and (8) cos BC = cos (AB - e AC). By virtue of (2), Df 5 and Df 4 we have (9) $\sin x = [\cos BC - \cos AB - \cos AC]\,[\cos^2 AC + \cos^2 BC - 2 \cos AB \cos AC \cos BC]^{-1/2}$ and (10) sgn cos AC = = sgn cos x. The conditions (9), (6), (1), (7) and (8) yield the equality (11) sin x = sin e AC, while (11), (10) and (8) entail (4) and (5). Finally, basing on the same definitions and theorems we can easily demonstrate the validity of (2) and (3) when (4) and (5) are assumed to be true.

T 35 $\bigwedge_{A,B,x} \{(0 < AB < \pi \wedge x \in (-\pi, \pi]) \Rightarrow$

$\Rightarrow \bigvee_{1C} [x = \xi_{AB}(C) \wedge w\ (A,B,C)]\}$

PROOF. For numbers $|x|$, AB there exists the sole number $b \in [0,\pi]$ such that (1) cos b = cos (AB - x). From (1) and L2 it follows the equality (2) $Q(AB, |x|, b) = 0$. By virtue of assumptions of our theorem T 35 we deduce, taking into account the existence of a unique point C satisfying the relations (3) $AC = |x|$ and (4) BC = b. Now, the desired thesis is a consequence of (4), (1) and T 34.

The above assures the correctness of the following Df 6 and the reasonableness of the consecutive T 36 :

Df 6 $\quad 0 < AB < \pi \Rightarrow \{C = \sigma_{AB}(x) \overset{df}{\Leftrightarrow} [x = \xi_{AB}(C) \wedge w(A,B,C)]\}$

T 36 $\quad 0 < AB < \pi \Rightarrow \bigwedge_{-\pi < x \leq \pi} \bigvee_{1C} C = \sigma_{AB}(x)$

We can also establish :

Df 7 $\quad 0 < AB < \pi \Rightarrow \varrho_{AB}(C) \overset{df}{=} \sigma_{AB}(\xi_{AB}(C)).$

As immediate corollaries from Df 7, T 32, T 36 and Df 6 we obtain :

T 37 $\quad 0 < AB < \pi \Rightarrow \bigwedge_C \bigvee_{1C'} C' = \varrho_{AB}(C)$

T 38 $\quad [0 < AB < \pi \wedge C' = \varrho_{AB}(C)] \Rightarrow w\,(A,B,C')$

T 39 $\quad [0 < AB < \pi \wedge C' = \varrho_{AB}(C)] \Rightarrow \xi_{AB}(C) = \xi_{AB}(C')$

T 40 $\quad [0 < AB < \pi \wedge x = \xi_{AB}(C)] \Rightarrow \begin{cases} AC \geq |x| & \text{for } AC \leq \pi/2 \\ AC \leq |x| & \text{for } AC \geq \pi/2 \end{cases}$

PROOF. In compliance with T30 we infer that $|\eta_{AB}(C)| \leq \sin AC$, and taking into account Df 5 we obtain the double inequality $|\sin x| \leq \sin AC$. This inequality entails $\sin|x| \leq \sin AC$. Since AC, $|x| \in [0,\pi]$ and since $\operatorname{sgn}\cos x = \operatorname{sgn}\cos AC$, then from the last inequality results the thesis of our theorem.

T 41 $\quad [(1)\ 0 < AB < \pi \wedge (2)\ w\,(A,B,C) \wedge (3)\ D' = \varrho_{AB}(D)] \Rightarrow \perp CD'D$.

PROOF. From (2) and T 15 results the existence of a point E such that (4) $AE = BE = CE = \pi/2$. Assume additionaly that (5) $D=E$. Now (4) and Df 3 entail the perpendicularity (6) $\perp BAE$ while (1), (3) and T 38 entail the collinearity (7) $w(A,B,D')$ of points A, B and D'. On the strength of (1), (3), T 39, (4), Df 4 and Df 5 we infer that the coordinates of points D and D' must vanish, namely (8) $\xi_{AB}(D) = \xi_{AB}(D') = 0$. From (1), (7), (8) by T34 and T8 we deduce that the points A and D' must coincide : $A = D'$. Conditions (1), (2), (6) and T 26 yield the perpendicularity (10) $\perp CAE$, from here taking into considerations (9) and (5) we obtain the promised thesis. The proof of the case $D = E^*$ is quite analogous, as under assumption (5). Let a subsequent additional assumption be (11) $E \neq D \neq E^*$. By using the theorem T22 we construct a point D_1 satisfying the condition (12) $ED_1 = \pi/2$ and (13) $w(E,D,D_1)$ The antipodal point D_1^* in accordance with T16 and T11 has also those properties. Without any loss of generality we can assume that $0 \leq DD_1 \leq \pi/2$. Conditions (12), (4) and Df 3 entail perpendicularities (15) $\perp ED_1C$ (16) $(\perp ED_1A \wedge \perp ED_1B)$. A consecutive perpendicularity (17) $\perp CD_1D$ follows from (12), (15), (13) and T26. Applying (12), (13), (16), Df 3 and T26 we obtain:

(18) $\cos AD_1 \cos DD_1 = \cos AD$ and (19) $\cos BD_1 \cos DD_1 = \cos BD$ while from (18) and (14) we infer (20) $\operatorname{sgn} \cos AD_1 = \operatorname{sgn} \cos AD$. The coordinates of points D and D_1 by virtue of (1), (18), (19), (20), Df 4 and Df 5 must coincide, so that (21) $\xi_{AB}(D) = \xi_{AB}(D_1)$. Taking into account (1), (3) and T39 we obtain the coincidence of coordinates of points D and D' too, viz. (22) $\xi_{AB}(D) = \xi_{AB}(D')$. Since (4) and (12) holds, then in compliance with T19 the relation (23) $w(A,B,D_1)$ is valid. From (1), (7), (21), (22), (23) by T35 we infer the identity of points $D_1 = D'$. This identity is in the presence of (17) equivalent to the thesis and thus the proof is completed.

T42 $[(1)\quad 0 < AB < \pi \wedge (2)\ CD < \pi/2 \wedge (3)\ w(A,B,C) \wedge$
$\wedge (4) \perp ACD] \Longrightarrow \varrho_{AB}(D) = C$.

PROOF. From (4) and Df 3 we infer (5) $\cos AC \cos CD = \cos AD$. Next, from (1), (3) and (4) by T 32 and Df 3 we obtain (6) $\cos BC \cos CD = \cos BD$. The theorem T32 and assumption (1) insures the existence of a unique number x such that the equality (7) $x = \xi_{AB}(D)$ holds. Bearing in mind (1), (2), (5) and (6) and applying Df 4 we infer (8) $\eta_{AB}(D) = \eta_{AB}(C)$. From (5) and (2) results also the equality (9) $\operatorname{sgn} \cos AD = \operatorname{sgn} \cos AC$. Conditions (1), (8), (9), (7) and Df 5 entail the equality (10) $x = \xi_{AB}(C)$. From relations (1), (10), (3) and

Df 6 we conclude that the equality (11) $C = \sigma_{AB}(x)$ holds. Finally as a consequence of (1), (11), (7) and Df 7 we obtain the required equality $C = \varrho_{AB}(D)$.

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Streszczenie

W artykule tym stanowiącym kontynuacje [5] w oparciu o aksjomaty A1, A2 i A3 zdefiniowano podstawowe pojęcia i udowodniono szereg twierdzeń geometrii metrycznej na sferze. Teorię rozwinięto w takim stopniu, aby móc udowodnić izomorfizm każdego modelu z modelem podstawowym S_0 . Dowód kategoryczności tej teorii będzie przedmiotem następnego artykułu.