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WŁODZIMIERZ ŚLĘZAK
WSP w Bydgoszczy

SOME CONTRIBUTIONS TO THE THEORY OF BOREL α SELECTORS

Let X and Y be topological spaces and let F be a map from X to $\mathcal{Y}$ the hyperspace of all nonempty subsets of Y. A continuous function f from X to Y is said to be a continuous selection for F if $f(x) \in F(x)$ for every $x \in X$. The notion of continuous selection was introduced and studied in detail by E. Michael, cf. [14-16] and the references there, Selection theorems are in obvious way generalizations of extension theorems. An extension problem is a selection problem in which F(x) is for every $x \in X$ either a singleton or the whole space Y. In the introduction of [14] Michael pointed out that the main result of his selection theory is the fact that most of the known extension theorems can be slightly changed and essentially generalized to suitable selection theorems. Let α be any countable ordinal number. Recall that a multifunction $F : X \to Y$ is said to be of lower class α if $F^{-}(V) := \{x \in X : F(x) \cap V \neq \emptyset\}$ is a Borel set in X of additive class α for each open set V in Y (cf. [11]). In [10] the following general problem is formulated: Under what assumptions can theorems known for $\alpha = 0$ be extended to arbitrary $\alpha < \Omega$?
Paper [4], see also [3], contains some finite-dimensional Borel α analogue of Michael's famous theorem 3.1''', see [14], p. 373 and 368. The aim of this work is to improve this theorem via so-called Castaing representation of F. We give also some rather trivial modifications of the results known in the case of measurable selectors (cf. [1], [2], [6], [8] [20]) and we include some facts from my recent works [18 - 19].

Following Rockafellar [17] we say $\{f_1, f_2, \ldots\}$ is a Castaing representation of F if each f_i is a selector for F, and $\{f_1(x), f_2(x), \ldots\}$ is dense in $F(x)$ for $x \in X$. Representations of this kind are exhibited by Castaing [1] for the case where the measurable structure on X is that of a Radon measure on locally compact space, and by Himmelberg [8] in the general situation where X is an abstract measurable space. The Castaing representation for multifunction of lower class α also exists by virtue of [13] . Since the theorem 4 in [13] is stated as a corrolary to some rather complicated results, we give here the direct proof of this theorem. Moreover, our theorem 1 gives the existence of denumerable sequence of selections provided that X is a perfectly normal topological space, while in [13] X is assumed to be a metric space. Let us recall that a normal topological space in which each closed set is a G_δ is called perfectly normal. The ordinal space $[0,\Omega]$ with topology generated by all sets of the form $\{x : x > \alpha\}$ and $\{x : x \leq \beta\}$ is an example of paracompact, hence also normal, topological space that is not perfectly normal. Namely the closed set $\{\Omega\}$ is not a G_δ . For, if $\{G_i ;\ i=1,2,\ldots\}$ is any countable collection of open sets containing Ω , then because the sets (α,β) are a basis, for each $i=1,2,\ldots$ there exists an ordinal $\alpha_i < \Omega$ such that $(\alpha_i, \Omega]$ is contained in G_i. Being countable, the collection $\{\alpha_i ;\ i=1,2,\ldots\}$ has an upper bound $\beta < \Omega$, so $\bigcap_{i=0}^{\infty} G_i \supset (\beta,\Omega] \neq \{\Omega\}$. Clearly $[0,\Omega]$ is paracompact. Indeed, let $\{U_i,\ i \in I\}$ be any open covering. Since the sets (λ,μ) form a basis, define $f: [0,\Omega] \to [0,\Omega]$ by associating with each $\beta \neq 0$ a $f(\beta) < \beta$ such that $(f(\beta),\beta]$ is contained in some U_i, and setting $f(0) = 0$. By induction, construct a sequence $\beta_0 = \Omega$, $\beta_1 = f(\Omega), \ldots, \beta_k = f(\beta_{k-1}), \ldots$ then $\beta_0 > \beta_1 > \ldots$ and, since every descending sequence of ordinals is finite, this terminates with some β_n . Because the process cannot be continued, $\beta_n = 0$, and so $(0,\Omega]$ is contained in $\bigcup_{k=1}^{n} (\beta_{k-1}, \beta_k]$. Choosing a U_{i_k} containing each $(\beta_{k-1}, \beta_k]$ and some $U_{i_o} \supset \{0\}$, we have a finite subcovering $\{U_{i_o}, U_{i_1}, \ldots, U_{i_n}\}^o$ of $\{U_i;\ i \in I\}$, which is consequently an open neighbourhood-finite refinement. Conversely the reader can verify that the subspace $[0,\Omega)$ is perfectly normal, but it is not paracompact: the open covering by sets $[0,\alpha)$, $0 < \alpha < \Omega$, has no open neighbourhood-finite refinement.

THEOREM 1. Let X be a perfectly normal topological space and let Z be a Polish space. Suppose that $f:X \longrightarrow Z$ is a multifunction with closed values. Then the following conditions are equivalent:

a) F is of lower class α, $\alpha > 0$

b) there exist Borel α functions $f_i:X \to Z$, $i=1,2,\ldots$ such that for each $x \in X$ we have $F(x)= \mathrm{Cl}\{f_i(x):\ i=1,2,\ldots\}$

P r o o f : (a) $\Longrightarrow$ b. Fix a complete metric d on Z. It suffices to show that for every $\varepsilon > 0$ there exist Borel α selectors $g_n:X \to Z$ for F such that $\{g_n(x):n=1,2,\ldots\}$ is an ε-net in $F(x)$ for each $x \in X$. Once this is done, we can put together the functions $\{g_n^k;\ n,k=1,2,\ldots\}$, to get the desired result, where g_n^k are Borel α selectors for F such that $\{g_n^k(x):\ n=1,2,\ldots\}$ is a k^{-1}-net in $F(x)$ for each $x \in X$. Since Z is separable, hence it is possible to cover Z by open balls $K(z_i, \varepsilon/2)$, $i=1,2,\ldots$ Put $X_i := \{x \in X:\ F(x) \cap K(z_i, \varepsilon/2) \neq \emptyset\}$, $i=1,2,\ldots$. Note that each X_i is of Borel additive class α, in virtue of the properties of F. Fix a Borel α selector $h:X \to Z$ for F. The existence of such a selector is ensured by famous theorem of Kuratowski and Ryll-Nardzewski, cf. [3], theorems 3 and 11. Suppose that X_i is nonempty, and define $F_i:X_i \longrightarrow Z$ by formula $F_i(x)= \mathrm{Cl}[F(x) \cap K(z_i, \varepsilon/2)]$. Notice that if G is open in Z, then
$F_i^-(G) := \{x \in X_i:\ F_i(x) \cap G \neq \emptyset\} = \{x \in X:\ \mathrm{Cl}[F(x) \cap K(z_i, \varepsilon/2) \cap G \neq \emptyset\} = \{x \in X:\ F(x) \cap G \cap K(z_i, \varepsilon/2) \neq \emptyset\}$ is of Borel additive class α in X. Since X_i is also of Borel additive class in X, hence $F_i^-(G)$ is of Borel additive class α of X_i

with the induced topology. We can therefore appeal once again to the Kuratowski and Ryll-Nardzewski theorem to get a Borel α selector $h_i : X_i \longrightarrow Z$ for F_i. There exist sets X_{ij}, $i,j=1,2,\ldots$, which are simultaneously of additive class less than α and of the multiplicative class less than α, such that for each $i=1,2,\ldots$ we have $X_i = \bigcup_{j=1}^{\infty} X_{ij}$.
For $j=1,2,\ldots$ define $h_{ij} : X \longrightarrow Z$ as follows

$$h_{ij} := \begin{cases} h_i & \text{on } X_{ij} \\ h & \text{on } X - X_{ij} \end{cases}$$

If X_i is empty, set $h_{ij}=h$ for every $j=1,2,\ldots$. In either case, as is easy to check, the functions h_{ij}, $j=1,2,\ldots$, are Borel α selectors for F. To conclude the proof of this item, we claim that for each $x \in X$, $\{h_{ij}(x);\ i,j=1,2,\ldots\}$ is an ε-net in $F(x)$. For, if $z \in F(x)$, we can find an i_o such that $z \in K(z_{i_o}, \varepsilon/2)$, hence $F(x) \cap K(z_{i_o}, \varepsilon/2)$ is nonempty and $x \in X_{i_o}$. So there is an j_o such that $z \in X_{i_o j_o}$. It follows that $h_{i_o j_o}(x) = h_{i_o}(x) \in F_{i_o}(x) \subset \mathrm{Cl}\ K(z_{i_o}, \varepsilon/2)$.
Consequently, $d(z, h_{i_o j_o}(x)) \leq d(z, z_{i_o}) + d(z_{i_o}, h_{i_o j_o}(x)) < \varepsilon$.
Thus $U := \{h_{ij};\ i,j=1,2,\ldots\ k=1,2,\ldots\} = \{u_1, u_2, \ldots\}$ is as required. b $\Longrightarrow$ a. Let G be an open set in Z and $U = \{f_1, f_2, \ldots\}$. Then $F^-(Z) := \{x : F(x) \cap G \neq \emptyset\} =$
$= \{x : [\mathrm{Cl}\ U(x)] \cap G \neq \emptyset\} = \{x : U(x) \cap G \neq \emptyset\} =$
$= \{x : u_i(x) \in G \text{ for some } u_i \in U\} = \bigcup_{n=1}^{\infty} \{x : u_n(x) \in G\} =$
$= \bigcup_{n=1}^{\infty} u_n^{-1}(G)$ is of Borel additive class α, so that F is of lower class α.
Now, let Z be a linear space. The convex hull and closed convex hull of a set $B \subset Z$ are denoted by conv B and Cl conv B, respectively. The following theorem is an easy generalization of theorem from [1], section 6 :

THEOREM 2. Let X be a perfectly normal topological space and let Z be a separable Frechet space and $F : X \longrightarrow Z$ a multifunction of lower class α with closed values. Then the multifunctions conv F , Cl conv F defined by $x \mapsto \mathrm{conv}\, F(x)$, $x \mapsto \mathrm{Cl}\,[\mathrm{conv}\, F(x)]$ are also of lower class α . Moreover, Z need only be a separable metric locally convex space, if each F(x) is assumed to be complete.

P r o o f : By theorem 1 there is a countable collection $U = \{u_1, u_2, \ldots, u_n, \ldots\}$ of Borel α selectors for F such that $F(x) = \mathrm{Cl}\, U(x)$ for all $x \in X$. Let Q be the set of all sequences $(q_1, q_2, \ldots, q_n, \ldots)$ of non-negative rational numbers such that all but finitely many q_n's are 0 and $\sum_{n=1}^{\infty} q_n = 1$.

The set Q is clearly countable and so is

$V := \{ \sum_{n=1}^{\infty} q_n \cdot u_n ;\ (q_1, q_2, \ldots, q_n) \in Q \}$.

V is a countable collection of Borel α functions such that $\mathrm{Cl}\, V(x) := \mathrm{Cl} \{ \sum_{n=1}^{\infty} q_n \cdot u_n(x) ;\ (q_1, q_2, \ldots) \in Q \} = \mathrm{Cl}\,\mathrm{conv}\, U(x) =$

$= \mathrm{conv}\, U(x)$ for all $x \in X$. Hence, again applying Theorem 1, conv F and Cl conv F are of lower class α .

THEOREM 3. Let X be a perfectly normal topological space and Z a separable Banach space. Let $f : X \to Z$ be a Borel α map and $r : X \to R$ a Borel α function with nonnegative real values. Then the multifunction defined by formula

$$F(x) = \bar{K}\,(f(x), r(x) := \{ z \in Z : \| f(x) - z \| \le r(x) \}$$

is of lower class α .

P r o o f. Let $(z_1, z_2, \ldots)$ be a dense sequence in the unit ball of Z. Put $u_n(x) := f(x) + r(x) \cdot z_n$, $n = 1, 2, \ldots$. Then each u_n is clearly of Borel class α and we have an equality

$$F(x) := \bar{K}\,(f(x), r(x)) = \mathrm{Cl}\ \{ u_n(x) ;\ n = 1, 2, \ldots \}$$

By theorem 1, F is of lower class α .

Using theorem 1 we are able to prove the following superposition theorem for multifunctions of lower class α :

THEOREM 4. Let X, Y and Z be separable metric spaces and $f : X \times Y \to Z$ a continuous function. Let $f : X \to Y$ be a multifunction of lower class $\alpha > 0$ with complete values. The the multifunction $G : X \to Z$ defined by formula

$G(x) = f(\{x\} \times F(x))$ is also of lower class α .

P r o o f : Applying the Castaing's representation, let U be a countable set of Borel α selectors for F such that $Cl\ U(x) = F(x)$ for each $x \in X$. Let B be an open subset of Z. Then we have $G^{-}(B) = \{x : f(\{x\} \times F(x)) \cap B \neq \emptyset\} =$
$= \{x : f(\{x\} \times Cl\ U(x)) \cap B \neq \emptyset\} =$
$\{X \ni x : f(x,y) \in B$ for some $y \in Cl\ U(x)\} =$
$\{x : f(x,u(x)) \in B$ for some $u \in U\} = \bigcup_{u \in U} \{x : f(x,u(x)) \in B\}$.
So it remains to show that $\{x : f(x,u(x)) \in B\}$ is a member of Borel additive class α. By separability of $X \times Y$ the map $h_u : X \to X \times Y$ defined by $h_u(x) = (x, u(x))$ is clearly of Borel class α. Then $\{x : f(x, u(x)) \in B\} = h_u^{-1}(f^{-1}(B))$ belongs to Borel additive class α, so that G is of lower class α.

Now, let Z be a Frechet space. If K is a closed, convex subset of Z, then a supporting set of K is, by definition, (see [14]), a closed, convex, proper subset S of K (S may even be a singleton) such that if an interior point of a segment in K belongs to S, then the whole segment is contained in S. The set $I(K)$ of all elements of K which are not in any supporting set of K will be called the inside of K. The family

$$D(Z) = \{B \subset Z : B = \text{conv}\ B \text{ and } B \supset I(Cl\ B)\}$$

introduced in [14] is seemingly the adequate range space for the Ceder-Levi theorem. The reader is refered to the excellent monograph [5] for the study of properties of convex sets.

Proposition 1.([14]) Every convex subset K of Frechet space Z which is either closed, or has an interior point or is finite-dimensional, belongs to $D(Z)$.

P r o o f. If K is closed, this is obvious. If K has an interior point, and if $z \in (Cl\ K) - K$ (complement of K in $Cl\ K$) then the Hahn-Banach theorem guarantees the existence of closed hyperplane $H \subset Z$ which supports $Cl\ K$ at z, but does not contain $Cl\ K$; clearly $H \cap Cl\ K$ is a supporting set of $Cl\ K$ and hence z belongs no to $I(Cl\ K)$. Finally, if K is finite dimensional, then K has an interior point with respect to the smallest linear variety $Z_1 \leq Z$ containing K. Hence $K \in D(Z_1) \subset D(Z)$.

Proposition 2. ([14]) If $\{t_1, t_2, \dots\}$ is a dense subset of a nonempty, closed, convex, separable subset K of a Frechèt space Z, and if

$$z_i = t_1 + \frac{t_i - t_1}{\max(1, d(t_i, t_1))}, \quad i=1,2,\dots$$

where d is an invariant metric on Z, then $z := \sum_{j=1}^{\infty} 2^{-j} z_j$ belongs to the inside I(K) of K.

P r o o f. Suppose $z \notin I(K)$. Then there exists a supporting set $S \subset K$ such that $z \in S$. Now for every $i=1,2,\dots, z$ is either an interior point of a segment in K one of whose end points is z_i, or else $z = z_i$. In either case we must have $z_i \in S$. But, for every $i=1,2,\dots$ the point z_i is either an interior point of the segment $[t_1 ; t_i] := \{t \in Z : t = at_1 + (1-a)t_i ; 0 \leq a \leq 1\}$ or else $z_i = t_i$, so in either case we must have $t_i \in S$. But $\{t_1, t_2, \dots\}$ is dense in K, and since S is closed, this finally implies that S=K, which is impossible.

We are now in a position to state and prove our main selection theorem:

THEOREM 5. Let X be a perfectly normal topological space and Z a separable Freohèt space. If $F : X \to Z$ is a multifunction of lower class α, $\alpha \geq 0$, whose values are in D(Z), then F has a Borel α selector.

REMARK 1. In case $\alpha = 0$ we must observe that X is countably paracompact.

P r o o f. Define $\mathrm{Cl}\, F : X \to Z$ by formula $(\mathrm{Cl}\, F)(x) = \mathrm{Cl}\,[F(x)]$; what we must find is a Borel α selector $f : X \to Z$ such that $f(x) \in I(\mathrm{Cl}\, F(x))$ for every $x \in X$. Obviously ClF is also of lower class α, in virtue of our theorem 2. Thus, in virtue of the theorem 1, Cl F has a Castaing representation $\{f_1, f_2, \dots\}$.

Now, let

$$g_i(x) = f_1(x) + \frac{f_i(x) - f_1(x)}{\max(1, d(f_i(x), f_1(x))}, \quad i=1,2,\ldots$$

where d denote an invariant metric on Z.
Put $f(x) = \sum_{j=1}^{\infty} 2^{-j} g_j(x)$. By Proposition 2 we have $f(x) \in I(\mathrm{Cl}\, F(x)) \subset F(x)$ since $F(x)$ belongs to $D(Z)$. On the other hand, since g_j, $j=1,2,\ldots$ are of Borel class α and since the series defining $f: X \to Z$ converges almost uniformly on X, it follows that f is also of Borel class α, and thus has all the required properties.

REMARK 2. Some particular values of F may fail to be convex, cf. [15]

REMARK 3. It seems to be possible to formulate the above theorem 5 in the same general framework as the one in which the selection theorem of Magerl [12] is formulated, i.e. in the framework of (k,a)-paracompact paved spaces and a-convex hull-operators on k-bounded uniform spaces.

REMARK 4. We may ask whether or not the algebraic structure on Z is essential in Ceder-Levy theorem. We give to formulate some open problem in this direction (cf. [19])

Let Y be an arbitrary topological space. A subset $B \subset Y$ is said to be relatively quasiclosed, if $B \supset \mathrm{Int}_{\mathrm{Cl}\, B}\, \mathrm{Cl}\, B$, where $\mathrm{Int}_{\mathrm{Cl}\, B}$ denotes the interior operation with respect to the topology induced on subspace Cl B of the space Y.

The following may be a promising program:

Problem: Let X be a perfectly normal space and let Y be a Polish space. Assume that $F: X \to Y$ is a multifunction of lower class α, $\alpha > 0$, with relatively quasiclosed values. Is it true, that this multifunction possesses always a Borel α selector ?

REMARK 5. (about relative topology) In "Topology" by J. Dungundji, p. 77_2 and also in "Introduction to topology" by H. Patkowska, p. 21_{20} the following false formula is errorously stated: $\mathrm{Int}_B A = B \cap \mathrm{Int}\, A$.

THEOREM 6. Let X be any metric space having nonmeasurable (in Borel sense) subset $A \subset X$ and let Y be an arbitrary infinite-dimensional Banach space. Then there exists a lower semicontinuous multifunction $F : X \to Y$ with convex and disjoint values but with no Borel measurable selector.

P r o o f : Let $\{e_i, i \in I\}$ be some Hamel basis for Y. Fix some index i_o and put $H := \bigoplus_{j \in J} \text{Span } e_j$, where $J = I - \{i_o\}$, and Span denote here linear hull operator. Next observe that H is a dense linear subspace of Y, and thus is convex. Let us define $F: X \to Y$ by formula

$$F(x) := \begin{cases} H & \text{if } x \in A \\ H + e_{i_o} & \text{if } x \in X - A . \end{cases}$$

It is easy to check that $F: X \to Y$ is lower semicontinuous. Indeed, for each open ball $B(y,r) = y + B(0, r)$ we have $F^-(B(y,r)) = X$ in virtue of density of values of F, both H and $H + e_{i_o}$. Let $f: X \to Y$ be an arbitrary selector for our multifunction F. We have $(e^*_{i_o} f)^{-1}(R - \{0\}) = X - A$ while $R - \{0\}$ is open. Since $X - A$ is not Borel measurable, hence the above equality means, that f is not weakly Borel measurable and thus it is not strongly Borel measurable. Obviously we have $H + e_{i_o} = \text{conv}(H + e_{i_o})$ and we observe that the intersection $(H^o + e_{i_o}) \cap H$ is empty. Thus our theorem is proved.

REMARK 6. In the case when our Banach space Y is also separable the values of F in the above theorem may be choosen to be an F_σ-sets. Indeed, let H in formula /§/ be a dense subspace of Y spanned by a countable familly $\{e_1, e_2, \dots\}$. Since $H = \bigcup_{n=1}^{\infty} H_n$, where $H_1 := \text{Span } e_1$, $H_n := H_{n-1} \oplus \text{Span } e_n$, and each H_n is closed as a finite-dimensional subspace of Y, hence we obtain that H belongs to the paving of F_σ-subsets of Y. A modification of this construction were used in my paper [18] to solving some open problem posed by J. Ceder and S. Levi in [4] in connection with his finite-dimensional version of our theorem 5. Note also, that our multifunction F given by formula /§/ is not of the form $F = g^{-1}$ for some singlevalued function $g : F(X) \to X$, where the image $F(x)$ is defined in obvious way as $F(X) := \bigcup_{x \in X} F(x)$.

REMARK 7. There exists a nonseparable prehilbert space Y and a multifunction F from the real line R to the one-dimensional open oonvex subsets of Y such that F is in lower class 1, but having no any Borel measurable selector. Indeed, define

$Y := \{ h : R \to R : \text{supp } h := \{x : h(x) \neq 0\} \text{ is finite}\}$.

then $\langle g | h \rangle := \sum_{x \in R} g(x) \cdot h(x)$ is a scalar product in Y.

Decompose R into 2 disjoint nonmeasurable subsets : $R = A \cup B$ and put

$$F(t) = \begin{cases} \{g \in Y : g(t) > 0 \text{ and supp } g = \{t\}\} & \text{if } t \in A \\ \{g \in Y : g(t) < 0 \text{ and supp } g = \{t\}\} & \text{if } t \in B . \end{cases}$$

See example 2 of [18] for the proof, that F is as required. Paper [6] contains many others interesting counterexamples.

THEOREM 7. Let C denote the space of complex numbers, and R the space of real numbers. There exists a lower semicontinuous multifunction $F : R \to C$ with arcwise connected values, but having no any Borel measurable selector.

P r o o f : Let S^1 denote the unit circle in the complex plane. Since the class of all Borel measurable functions from R to S^1 has the same cardinality as R it is clear that we can choose a function $g : R \to S^1$ such that the graph of f intersects the graph of each Borel measurable function from R to S^1. Put $F(x) := S^1 - \{g(x)\}$ and observe that $F^{-}(U)$ is empty or the whole space R whenever U is open in C. Thus F is lower semicontinuous. Obviously the values of F are open arcs. We prove that F is without any Borel selector. Assume ad absurdum that $f : R \to S^1$ is some Borel selector for F. There is a point $x_0 \in R$ such that $f(x_0) = g(x_0)$ hence $f(x_0)$ not belongs to $F(x_0) = S^1 - \{g(x_0)\} = S^1 - \{f(x_0)\}$. This contradiction finish our argument.

THEOREM 8. There exists a lower semicontinuous multifunction F from the real line R to the complex plane C with open arcs as values and $G_{\delta\sigma}$ graph but with no Borel 1 selector.

P r o o f : (cf. [18], ex. 4) Recently Z. Grande exhibited a Borel 2 function $g : [0, 2\pi) \to [0,2\pi)$ which intersects each Borel 1 function from $[0,2\pi)$ into $[0,2\pi)$, see [7]. His construction based upon Kantorowicz universal function K for Borel 1 functions, namely $g(x) := K(x,x)$. By lifting theorem, we may assert that $x \mapsto \hat{g}(x) := \exp i \cdot g(x) = \cos g(x) + i \sin g(x) \in S^{-1}$ is a function from $[0, 2\pi)$ into the unit circle S^1, intersecting each Borel 1 function $f : [0,2\pi) \to S^1$. Let us define our multifunction $F: R \to C$ by formula $F(x) := S^1 - \{\hat{g}([x])\}$, where $[x]$ is the class of $x \in R$ in the quotient group $R / 2\pi Z \cong [0,2\pi)$. It is easy to check as in theorem 7 that F is lower semicontinuous with open arcs as values but admitting no Borel 1 selector. Moreover, since $\hat{g}$ belongs also to the second Borel class and thus $Gr\, \hat{g} := \{(x,y) : y = \hat{g}(x)\}$ is a $F_{\sigma\delta}$ subset of the cylinder
$[0,2\pi) \times S^1$, hence $Gr\, F := \{(x,y) : y \in F(x)\} = [0,2\pi) \times S^1 - Gr\, \hat{g}$ is a $G_{\delta\sigma}$- subset of this cylinder and thus of the whole product space $R \times S^1$. This completes our argument.
From theorems 7 and 8 it follows that the range space in our theorem 5 cannot be generalized in various ways. Another interesting questions is whether of not the polishness of Z is essential in theorem 1. We use the example 5 of [18] to answer the above problem.

THEOREM 9. Assuming the continuum hypothesis, there exist metric spaces X and Z and a multifunction $F: X \to Z$ with nonempty closed values such that F is in first lower class, but F has no Borel selector.

P r o o f : (cf. [9], [18]) Let Ω denote the smallest countable ordinal. Let $Z = \{\alpha : \alpha < \Omega\}$ denote the set of all countable ordinals with the discrete topology, while the real line $R \approx X$ is inquired together with usual euclidean topology. Arrange all points of X in a transfinite sequence of type Ω and define the multifunction $F:X \to Z$ by formula

$$X \ni x_\alpha \longmapsto \{\beta : \beta \geq \alpha\} \qquad \text{for } \alpha < \Omega .$$

Evidently F is of first lower class. Let $f:X \to Z$ be any selector for F. Since $f(x_\alpha) \geq \alpha$ for every $\alpha < \Omega$, each fiber $f^{-1}(\zeta)$, $\zeta < \Omega$ is countable. Since the fibers are pairwise disjoint and since the family of all subsets of all fibers has cardinality greater than the set of all Borel sets, it follows that there exists some subsets Z_o of Z so that $f^{-1}(Z_o) = \bigcup_{\zeta \in Z_o} f^{-1}(\zeta)$ is not B-measurable. Since Z_o is both open and closed in the discrete space Z, this completes the argument.

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PRZYCZYNEK DO TEORII SELEKTOROW BORELOWSKIEJ KLASY ALFA

Streszczenie

Głównym wynikiem pracy jest następujące uogólnienie twierdzenia Cedera i Leviego: każda multifunkcja F:X → Z, gdzie X jest doskonale normalną przestrzenią topologiczną, a Z ośrodkową przestrzenią Frechèta, przyjmująca swoje wartości z określonej rodziny D/Z/ i będąca dolnej klasy alfa, posiada borelowski selektor klasy alfa. Liczne kontr-przykłady zamieszczone w pracy wskazują, że przyjęte ograniczenia na wartości F nie mogą być osłabione.