ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ W BYDGOSZCZY
Problemy Matematyczne 1983/84 z.5/6

WŁODZIMIERZ ŚLĘZAK
WSP w Bydgoszczy

MULTIFUNCTIONS OF TWO VARIABLES WITH SEMICONTINUOUS SECTIONS

A multifunction $F:T \rightarrow Z$ is a function whose value $F(t)$ for $t \in T$ is a non-empty subset of Z. Given topological structures on T and Z (or even convergence structures) it is possible to define continuity of F in various ways (see e.g. [1, 12, 19]).

Definition 1. We say F that is upper semicontinuous at point $t_o \in T$ if for each open set G containing $F(t_o)$ the set

$$F^+(G) := \{t \in T : F(T) \subset G\}$$

is open. Dually, we say that F is lower semicontinuous at point $t_o \in T$ if for any open set G which meets $F(t_o)$ the set

$$F^-(G) := \{t \in T : F(t) \cap G \neq \emptyset\} = T - F^+(Z-G)$$

is open in T. F is called continuous at point $t_o \in T$ if it is both upper and lower semicontinuous at t_o.
F is called upper (resp. lower) semicontinuous if it is upper (resp. lower) semicontinuous at each point of T. When F is compact-values and lower semicontinuous then the set of all points of upper semicontinuity of F is the complement of the union of any countable family of nowhere dense closed sets. If F is closed-valued and upper semicontinuous then the set of its points of lower semicontinuity is the complement of some countable union of nowhere dense closed sets (see [18]). We may ask the following problem related to the Shih Shu-Chung [18] results:
Problem 1. Let X be a Baire space, i.e. the intersection of each countable family of open dense sets in X is dense. Let A and B be two disjoint F_σ-sets of the first category

in X. Does there exist a function f from X into R with positive values such that f is lower semicontinuous exactly on X-A, upper semicontinuous exactly on X-B and continuous on X-A-B. In some particular cases the answer is affirmative in virtue of some recent works of Z. Grande [8] and T. Natkaniec. If such a function f exists, then the multifunction $F:X \longrightarrow R$ defined by formula $F(x)=[-f(x);\ f(x)]$ is upper semicontinuous exactly on X-B, lower semicontinuous exactly on a preassigned subset $X-A \subset X$, and has compact, convex values.

Let T=XxY with the cartesian product topology, i.e. the smallest topology for which all projections are continuous. If S is any subset of XxY and x is any point of X, we shall call the set $S_x := \{y:(x,y)\in S\}$ a section of S, or, more precisely, the section determined by x. At times when it is important to call attention not so much to the particular point which determines the section as merely to the fact that the section is determined by some point of the space X we shall use the phrase X-section. The main point is to distinguish such a section from a Y-section determined by a point y in Y ; the latter is defined , of course, as the set $S^y := \{x:\ (x,y)\in S\}$. We emphasize that a section of a set in a product space is not a set in that product space but a subset of one of the component spaces. If F is any multifunction defined on a subset S of the product space $X \times Y$ and x is any point of X, we shall call the multifunction F_x, defined on the section S_x by formula $F_x(y)= F(x,y)$, a section of F, or, more precisely an X-section of F, or, still more precisely, the section determined by x. The concept of a Y-section of F, determined by a point y in Y is defined similarly by $F^y(x)= F(x,y)$. Notice, that every section of a lower (resp. upper) semicontinuous multifunction is a lower (resp. upper) semicontinuous multifunction. Multifunctions of two variables have been studied by C. Castaing [2,3] , A. Cellina [4] ; A. Fryszkowski [6], T. Neubrunn [14], and B. Ricceri [15,16].

The purpose of this paper is to obtain some multivalued analogue of the renowned Kempisty theorem (see [9]). We start with the following lemma:

Lemma 1. Let (X,d_X) and (Y,d_y) be a metric spaces, Z an arbitrary topological space and $F:X\times Y \to Z$ a multifunction with lower semicontinuous Y-sections, not necessarily closed--valued. Then for each real constant $r>0$, the multifunction $F_r:X\times Y \to Z$ defined by formula

$$/\S/ \quad X\times Y \ni (x_o,y_o) \mapsto F_r(x_o,y_o) = \bigcup_{y\in K(y_o,r)} F(x_o,y) \subset Z$$

is lower semicontinuous on the whole space $X\times Y$.
$K(y_o,r) := \{y\in Y: d_y(y,y_o)<r\}$ means here the d_Y-ball of radius r and center y_o.

P r o o f. Let $(x_o,y_o)\in X\times Y$ be an arbitrary point.
We shall show that the multifunction F_r is lower semicontinuous at point (x_o,y_o). Let $G\subset Z$ be an open set for which $F_r(x_o,y_o)\cap G \neq \emptyset$. By /§/ there exist a point $y_1\in K(y_o,r)$ such that $F(x_o,y_1)\cap G$ is nonempty. Since the section F^{y_1} is lower semicontinuous on X, hence there exists a radius $r_1>0$ such that for each $x\in K(x_o,r_1)$ the intersection $F(x,y_1)\cap G$ is nonempty. Since $d_Y(y_o,y_1)<r$, there exists a $r_2>0$ such that $d_Y(y_o,y_1)= r - r_2$. By a triangle inequality, we have $d_Y(y_1,y)\leq d_Y(y_1,y_o)+ d_Y(y_o,y)<(r-r_2)+r_2=r$ for each $y\in K(y_o,r)$. Thus y_1 belongs to the ball $K(y,r)$ for every $y\in K(y_o,r_2)$. Consequently the inclusion

$$F(x,y_1)\subset \bigcup_{y_3\in K(y,r)} F(x,y_3)= F_r(x,y)$$

is valid whenever $y\in K(y_o,r_2)$ and $x\in K(x_o,r_1)$. But $F(x,y_1)\cap G\neq\emptyset$ so $F_r(x,y)\cap G$ is nonempty too, whenever $(x,y)\in K(x_o,r_1)\times K(y_o,r_2)$. The set $V(x_o,y_o)= K(x_o,r_1)\times K(y_o,r_2)$ is an open neighbourhood of an $(x_o,y_o)\in X\times Y$. The lower semicontinuity of F_r at (x_o,y_o) is thus proved.
The following two lemmas are generally known, but they are included for the sake of completeness.

<u>Lemma 2.</u> Let X,Y and Z be as before, and let $H: X\times Y \to Z$ be a lower semicontinuous multifunction with arbitrary values. Then the multifunction $\bar{H}: X\times Y \to Z$, defined by $\bar{H}(x,y) := \overline{H(x,y)}$ is also lower semicontinuous. The dash stands here for the closure operation in Z.

P r o o f. Let (x_o,y_o) be an arbitrary point of $X\times Y$, and let $G\subset Z$ be an open set for which $\bar{H}(x_o,y_o)\cap G$ is nonempty. Since $H(x_o,y_o)$ is dense in $\bar{H}(x_o,y_o)$, so also $H(x_o,y_o)\cap G$ is nonempty. By lower semicontinuity of the multifunction G there exists a neighbourhood $V(x_o,y_o)\subset X\times Y$ such that $H(x,y)\cap G$ is nonempty whenever $(x,y)\in V(x_o,y_o)$.

By inclusion $H(x,y)\subset \bar{H}(x,y)$ the intersection $\bar{H}(x,y)\cap G$ is nonempty too. Thus $\bar{H}$ is lower semicontinuous at (x_o,y_o)..

<u>Lemma 3.</u> Let X,Y be as before and let Z be a regular space , i.e. each point $z\in Z$ and each closed set $D\subset Z$ not containing z, have disjoint neighborhoods. If a multifunction $F: X\times Y \to Z$ with closed values has upper semicontinuous X-sections then the inequality

$$F(x_o,y_o) = \limsup_{x \to x_o} F(x,y_o) := \bigcap_{r^{-1}=1}^{\infty} \bar{F}_r(x,y_o)$$

holds whenever $(x_o,y_o)\in X\times Y$.

P r o o f. $\supset$: Let z belongs to $Z-F(x_o,y_o)$. Then there exist an open set $G\supset F(x_o,y_o)$ and an open neighborhood $W(z)\subset Z$ of a z such that $W(z)\cap G = \emptyset$.

The section F_{x_o} being upper semicontinuous at the point $y_o\in Y$, by definition there exists a radius $r_3 > 0$ such that $y\in K(y_o,r_3)$ implies $F(x_o,y)\subset G$. Thus $W(z)\cap F(x_o,y)$ is empty for $y\in K(y_o,r_3)$. Then the intersection $W(z)\cap F_{r_3}(x_o,y_o)$ is empty, too. This means that z is no cluster point of $F_{r_3}(r_o,y_o)$, and thereby the intersection $\bar{F}_{r_3}(x_o,y_o)\cap W(z)$ is empty .

Therefore $z\notin \limsup_{x\to x_o} F(x,y_o)\subset \bar{F}_{r_3}(x_o,y_o)$ and

$$\limsup_{x \to x_o} F(x,y_o)\subset F(x_o,y_o).$$

$\subset$: If $z\in F(x_o,y_o)$ then for each radius $r>0$, a point z

belongs to $F_r(x_0,y_0) \supset F(x_0,y_0)$. Consequently $z \in \limsup_{x \to x_0} F(x,y_0)$ and thereby $F(x_0,y_0) \subset \lim_{x \to x_0} F(x,y_0)$.
The proof is complete.

REMARK 1. This proof is very similar to the proof of proposition 1.4. in [12]. Notice, that [12] contains a typographical error in 144_{12}.

We are now in a position to state and prove our main theorem. Recall that a multifunction $F: T \to Z$ is of upper class α if $F^+(G)$ is a Borel set in T of additive class α for each open set G in Z Dually, we say F that is of lower class α if the inverse image $F^-(G)$ of each set G open in Z is a Borel set of additive class α in T.

THEOREM 1. Let $F: X \times Y \to Z$ where X and Y are metric spaces and Z is a compact metrizable space. If F is closed--valued multifunction with lower semicontinuous Y-sections and upper semicontinuous X-sections, then F is of upper class 1.

P r o o f: Lemmas 1,2 and 3 imply immediately that F is countable intersection of family $\{\overline{F}_r : r = 1, 2^{-1}, 3^{-1}, \ldots\}$ of lower semicontinuous multifunctions with closed values. Each F_r is of upper class 1, in fact it is of first Baire class as function of $X \times Y$ into the hyperspace of all closed nonempty subsets of Z with exponential (Vietoris)topology. This follows from theorems 1 and 2 of reference [11].
But if $\overline{F}_r$ is in upper class α for $r = 1, 2^{-1}, 3^{-1}, \ldots$, then so is $\overline{F}_1 \cap \overline{F}_{2^{-1}} \cap \overline{F}_{3^{-1}} \cap \ldots$ (see [11] , th.4 and [10] for the proof). Hence F is also in upper class 1 and our theorem 1 is proved.

REMARK 2. Let R denote the real line, and let $g, h : X \times Y \to R \cup \{-\infty, +\infty\}$ be two extended real-valued functions, such that $g(x,y) \leq h(x,y)$ for every (x,y) in $X \times Y$. Define $F: X \times Y \to R$ by $F(x,y) = \{z \in R : g(x,y) \leq z \leq h(x,y)\}$. F is of lower (resp. upper) class α if and only if g is of upper (resp. lower) class α and h is of lower (resp. upper) class α in the sense of W.H. Young. Thus our theorem 1 can be deduced from the famous Kempisty theorem [9] in the case Z=R. Somewhat conversely we may ask the following open problem:

Problem 2. Do there exist metric spaces X, Y and Z and a multifunction $F:X\times Y \to Z$ with closed values, lower semicontinuous Y-sections and upper semicontinuous X-sections such that F is not in lower class 1 ? x) In the sequel we illustrate some bad behaviour of multifunctions whose X-sections and Y-sections are simultaneously lower semicontinuous or upper semicontinuous.

THEOREM 2. There exists a multifunction $F:R\times R \to R$ with compact convex values, whose X-sections and Y-sections are lower semicontinuous and fails to be continuous at no more that one point, but F is not in any upper or lower Borel class.

E x a m p l e: Let S be a nonmeasurable Sierpiński set whose X-sections and Y-sections are singletons (see [17] for the construction) Define $F(x,y)=[-1;1]$ if $(x,y)\in S$ and $F(x,y)=[-3;3]$ otherwise. Clearly $F_x(y)=[-1;1]$ if $S_x=\{y\}$ and $F_x(y)=[-3;3]$ if $S_x\neq\{y\}$. Obviously each section $F_x:R\to R$ is lower semicontinuous, since $F_x^-((a,b))$ is empty or whole plane R^2 for open intervals (a,b). Symmetricaly each F^y is lower semicontinuous. The values of F are clearly convex and compact. But the inverse image of the open interval $(2;4)$ and of the closed set $\{3\}$ under F is not Borel :

$$F^-((2;4))=F^-(\{3\})=R^2-S$$

REMARK 3. The above example also show, that there exists a multifunction $F:R\times R\to R$ with convex compact values, whose X-sections are of lower class 1 and Y-sections are of upper class 1, but F is not in upper class 2. Then the theorem 1

---

x) This problem has a negative answer in case where Z is separable . Proof will appear in a later paper.

cannot be generalized for higher classes.

REMARK 4. Somewhat surprisingly, theorem 2 also show, that Carathéodory's condition, i.e. lower semicontinuity of X-sections and measurability of Y-sections is not sufficient that the multifunction $F:X\times Y \to Z$ be measurable nor weakly measurable (see definition 3 below)

THEOREM 3. There exist a nonmeasurable multifunction $F:R^2 \to R$ with compact, convex values and upper semicontinuous all sections.

E x a m p l e : Put $F(x,y)=[-3;3]$ if $(x,y)\in S$ and $F(x,y)=[-1;1]$ otherwise. It is easy to check, that F is as required.

THEOREM 4. There exists a nonmeasurable (nor weakly measurable) multifunction $F:R\times R \to R'$ whose values are G convex sets, whose X-sections are lower semicontinuous and Y-sections are continuous (i.e. semicontinuous in both senses).

E x a m p l e : Let $C\subset R$ be a dense, border, nonmeasurable set, whose inner density at each point is 0, and outer density at each point is 1. Put $F(x,y)=[0;1]$ if $y\in C$ and $F(x,y)=[0;1)$ otherwise. It is easy to check, that F is as required.

Definition 2 (cf.[8]) Let T be second countable topological space and Z a metric space. Assume that $F:T \to Z$ has closed values. We say, that F is strongly lower semicontinuous at point $t_o\in T$ if it is lower semicontinuous at this point and, moreover, there exists an open set $U\subset T$ such that $t_o\in \bar{U}$ and we have $\lim_{U\ni t\to t_o} F(t)=F(t_o)$, where the limit is assumed with respect to the Hausdorff metric

$$h(K_1,K_2)=\operatorname{arc\,tg}\max\left[\sup_{z_1\in K_1} \operatorname{dist}(z_1,K_2),\sup_{z_2\in K_2} \operatorname{dist}(z_2,K_1)\right]$$

where $\operatorname{dist}(z_o,K):=\inf\{d(z_o,z):z\in K\}$ and d denote the metric in Z.

REMARK 5. In the above definition we may also used another metric on the familly of all closed non-empty subsets of Z. Let p be a fixed point of Z. The metric h_p is defined by formula

$$h_p(K_1,K_2) := \sup_{z \in Z} \left\{ |\mathrm{dist}(z,K_1) - \mathrm{dist}(z,K_2)| \cdot \exp[-d(p,z)] \right\}$$

Definition 3. Let (T,S,m) be a measurable space and Z a topological space. We say, that a multifunction $F:T \to Z$ is weakly measurable if the set $F^-(G)$ is m-measurable for each open set G in Z.

THEOREM 5. Let X and Y be two locally compact topological spaces, with Radon measures m_X, m_Y respectively. Let Z be a locally compact metric space and let $F:X\times Y \to Z$ be a multifunction with closed values, such that all X-sections are strongly lower semicontinuous and all Y-sections are lower semicontinuous in common sense. Then F is $m_X \otimes m_Y$ - weakly measurable.

P r o o f : We argue by contradiction. Assume that F is not $m_X \otimes m_Y$ -weakly measurable. Then there exists an open set G in Z such that $F^-(G)$ is not $m := m_X \otimes m_Y$ -measurable. Hence there is a m-measurable subset $A \subset F^-(G)$ such that the difference $F^-(G) - A$ is simultaneously of interior measure m null and of exterior measure m positive :

$$m_*[F^-(G) - A] = 0 \wedge m^*[F^-(G) - A] > 0$$

By virtue of locally compactness of Z there is a increasing family $G_1 \subset \bar{G}_1 \subset G_2 \subset \bar{G}_2 \subset \dots \quad \dots \subset G$ of open sets in Z such that $F^-(G) = F^-\left(\bigcup_{i=1}^{\infty} G_i\right) = \bigcup_{i=1}^{\infty} F^-(G_i)$

We can select a suitable index i_0 such that $m^*[F^-(G_{i_0}) - A] > 0$

Define $B := F^-(G_{i_0}) - A$ and observe that for every subset $S \subset X\times Y$ such that $m^*(S \cap B) > 0$ we have also $m^*(S \cap F^+(Z-G)) > 0$. Since all Y-sections are lower semicontinuous, hence for every point $(x,y) \in B \subset X \times Y$ there correspond a basic open set $U(x,y)$ in X such that $x \in U(x,y)$ and $F(p,q) \cap G_{i_0}$ is nonempty whenever $p \in U(x,y)$. Since X satisfies the second axiom of countability and since $m^*(B)$ is positive, it follows that there is a basic open set $U_0 \subset X$ such that the exterior measure of the set

$$C := \{(x,y) \in B : U(x,y) = U_o\}$$

is positive , $m^*(C) > 0$. Define

$$D := \{y \in Y : \text{there is a point } x \in X \text{ such that } (x,y) \in C\}$$

and $E := \{q \in Y : \lim_{i \to \infty} \dfrac{m^*[D \cap \bar{A}_i^q]}{m[\bar{A}_i^q]} = 1\}$

when $A^q = \{A_1^q, A_2^q, \ldots\}$ is a suitable open filterbase (of differentiation) in Y convergent to singleton q.
Then the exterior measure $m_Y^*(D) = m_Y^*(D \cap E)$ is positive and the set E is m_Y-measurable in Y. Fix a point $y_o \in D \cap E$ and observe that $F(p,y_o) \cap G_{i_o}$ is nonempty for each $p \in U_o$. Since all X-sections of F are strongly lower semicontinuous, hence for each point

$$(x,y) \in [U_o \times (E \cap A_i^{y_o})] \cap F^+(Z - G)$$

there correspond a basic open set $V(x,y) \subset A_1^{y_o} \subset Y$ such that the inclusion $F(x,q) \subset Z - \bar{G}_{i_o+2}$ holds whenever $q \in V(x,y)$
Since Y satisfies the second axiom of countability and since the set $K_o = [U_o \times (E \cap A_i^{y_o})] \cap F^+(Z - G)$ is of the positive exterior measure m , there is a basic open set $V_1 \subset Y$ such that the set $K_1 = \{(x,y) \in K_o : V(x,y) = V_1\}$ has positive exterior measure $m^*(K_1) > 0$, too .

Define $M_1 = \{x \in X$: there is a point $y \in Y$ such that (x,y) belongs to $K_1\}$ and $H_1 = \bar{M}_1$ (closure with respect to X).
Let us observe that $m_X^*(G_1) > 0$ and $F(x,y) \subset Z - G_{i_o+2}$ for each point $(x,y) \in H_1 \times V_1$, in virtue of the lower semicontinuity of the Y-sections.
So far, we proceed by induction. For each point $(x,y) \in [H_1 \times (E \cap A_2^{y_o})] \cap F^+(Z - G)$ there exist a basic open set $V(x,y) \subset A_2^{y_o} \subset Y$ such that $F(x,q) \subset Z - G_{i_o+2}$ whenever $q \in V(x,y)$. Consequently there is a basic open set $V_2 \subset A_2^{y_o} \subset Y$ such that the set

$$K_2 = \{(x,y) \in [H_1 \times (E \cap A_2^{y_o})] \cap F^+(Z - G) : V(x,y) = V_2\}$$

is of the positive exterior measure $m_2^*(K_2) > 0$. Define

$M_2 = \{ x \in X :$ there is a $y \in Y$ such that $(x,y) \in K_2 \}$ and $H_2 = M_2$ and observe that $F(x,y) \subset Z - G_{i_0+2}$ for each point $(x,y) \in H_2 \times V_2$. Moreover $m_X^*(M_2) > 0$, $H_2 \subset H_1$ et $m_X^*(H_2) > 0$. Proceeding inductively, in the n-th steep we have a closed set $H_n \subset H_{n-1}$ of positive measure m_X and a basic open set $V_n \subset A_n^{y_0} \subset Y$ such that $F(x,y) \subset Z - G_{i_0+2}$ whenever $(x,y) \in H_n \times V_n$.

Let x_0 belongs to $\bigcap_{n=1}^{\infty} H_n$. Since $x_0 \in U_0$, hence we have $F(x_0,y_0) \cap G_{i_0} \neq \emptyset$. On the other hand y_0 belongs to $\overline{\bigcup_{n=1}^{\infty} V_n}$, and $F(x_0,y) \subset Z - G_{i_0+2}$ whenever $y \in \bigcup_{n=1}^{\infty} V_n$. It follows that F_{x_0} is not (strongly) lower semicontinuous at the point y_0, completing the proof. Recall that a function $f : T \to Z$ is said to be a selector for $F : T \to Z$ if $f(t)$ is a member of the set $F(t)$ for each $t \in T$. Based on the above definition we present a theorem inspired by papers [6] and [15-16]:

THEOREM 6. Let F be a multifunction from $X \times Y$ where X and Y are the separable locally compact metric spaces, onto closed and convex subsets of separable Banach space Z. We assume that all Y-sections of F are lower semicontinuous and that all X-sections of F are upper semicontinuous. Then there is a Borel 1 selector $f : X \times Y \to Z$ of F with continuous Y-sections and having X-sections of the first Baire class.

P r o o f: Let $C(X, Z)$ denote the space of all continuous maps by X into Z. The compact-open topology in $C(X,Z)$ is that having as subbasis all sets $\{ f \in C(X,Z) : f(K) \subset G \}$, where $K \subset X$ is compact and $G \subset Z$ is open. It is known, that under the assumption of theorem 6, $C(X,Z)$ is separable, locally convex linear topological space. Define the multifunction $P : Y \to C(X,Z)$ by formula

$P(y) = \{ f \in C(X,Z) : f(x) \in F(x,y)$ for each $x \in X \}$.

We prove that P is upper semicontinuous multifunction from Y into closed and convex subsets of linear topological space $C(X, Z)$. In fact, it suffices to prove that $P^-(D) \in F_\sigma(Y)$ where D is a closed set of the form

$$D := D(f_o, \varepsilon, K) := \{ f \in C(X,Z) : \sup_{x \in K} \| f(x) - f_o(x) \| < \varepsilon \}$$

where K is compact in X, f_o belongs to $C(X,Z)$ and $\varepsilon > 0$ is positive real. Denote by $\bar{K}(f_o(x), \varepsilon)$ the closed ball in Z centered at $f_o(x)$ and of radius ε. Then by celebrated Michael selection theorem [13] we have

$$P^-(D) = \{ y : P(y) \cap D(f_o, \varepsilon, K) \neq \emptyset \} =$$

$$= \{ y : \bar{K}(f_o(x), \varepsilon) \cap F(x,y) \neq \emptyset \text{ for each } x \in K \}$$

Since X-sections of F are upper semicontinuous, i.e. $F_x^-(B)$ is closed for each $B = \bar{B} \subset Z$, we have that the set

$$\{ y : F(x,y) \cap \bar{K}(f_o(x), \varepsilon) \neq \emptyset$$

is closed for each $x \in K$. Therefore

$$P^-(D) = \bigcap_{x \in K} \{ y : F(x,y) \cap \bar{K}(f_o(x), \varepsilon) \neq \emptyset \} = \bigcap_{x \in K} F_x^-(\bar{K}(f_o(x), \varepsilon))$$

is closed in Y, and consequently P is upper semicontinuous. Let us recall that locally compact separable metric space X is also σ-compact, i.e. it can be expressed as the union of at most countably many compact spaces. Write $X = \bigcup_{i=1}^{\infty} U_i$ where the U_i are open, $\bar{U}_i \subset U_{i+1}$ and U_i is relatively compact for each i. For all $f_1, f_2 \in C(X,Z)$ and each $n = 1, 2, \ldots$ define

$$r_n(f_1, f_2) := \min (n^{-1}, \sup \{ \| f_1(x) - f_2(x) \| ; x \in \bar{U}_n \})$$

Then $d(f_1, f_2) := \sup \{ r_n(f_1, f_2) ; n = 1, 2, \ldots \}$ metrizes the compact-open topology in $C(X,Z)$, but there is no metric d_1 in Z such that $d^+(f_1, f_2) := \sup \{ d_1(f_1(x), f_2(x)) : x \in X \}$ metrizes this topology.

Thus $C(X,Z)$ is a Polish space and hence there exists a Borel 1 selector p for our multifunction $P : Y \to C(X, Z)$ (see [5]).

Put $f(x,y) = p(y)(x)$. By [7] a map $f : X \times Y \to Z$ is in the first Baire class and thereby has the Baire 1 X-sections.

Moreover all Y-sections are clearly continuous, which completes the proof of our theorem 6.

REMARK 6. In [6] p. 43_{19}, A. Fryszkowski errorously treat C(X,Z) as a Banach space. Hence his proof of lemma in p.43 is not correct. But this proof can be easily improved using the above technique.

REMARK 7. Some particular values of the multifunction F in theorem 6 may fail to be convex (see [15]). Namely, if there exists a subset E of X, with $\dim_X E \leq 0$ such that F(x,y) is convex for each $(x,y) \in (X - E) \times Y$ only, then the theorem 6 is also true. The sign $\dim_X E \leq 0$ means here that $\dim B \leq 0$ for each $B \subset E$ which is closed in X, where dim B denoted the covering dimension of T.

I am indebted to Z. Grande for his constant willingness to discuss some details. His imaginative and penetrating criticisms and suggestions have let to many improvements.

REFERENCES

[1] Berge C., Espaces Topologiques-Fonctions Multivoques, Dunod, Paris 1959

[2] Castaing C., Sur l'existence des sections separement measurables et separement continues d'un multi-application. Exposé no 14; Seminaire d'Analyse Convexe, Montpellier 1975

[3] Castaing C., A propos de l'existence des sections séparément measurable et separement continues d'une multiapplication séparément measurable et séparément semi-continue inferieurement, Exposé no 6; Seminaire d'Analyse Convexe, Montpellier 1976

[4] Cellina A., A selection theorem, Rend. Sem. Mat. Univ. Padova, vol. 55 (1976), 143-149

[5] Engelking R., Selectors of the First Baire Class for Semicontinuous Set-valued Functions, Bull. de l'Acad. Polon. des Sciences, Ser. des sci. math., astr., phys., Vol. XVI, no 4 (1968), 277-282.

[6] Fryszkowski A., Carathéodory type Selectors of Set-valued

Maps of Two Variables, Ibid., vol. XXV, no 1 (1977)41-46
[7] Grande Z., Sur un probleme de Ricceri, Colloquium Math.
[8] Grande Z., Quelques remarques sur la semi-continuité superieure, Fundamenta Math.CXXV no 1 (1985), 1-13
[9] Kempisty S., Sur les fonctions semi-continues par rapport á chacune de deux variables, Fund. Math. 14 (1929) 237-241.
[10] Kuratowski K., Les fonctions semi-continues dans l'espace des ensembles fermés , Fund.Math. 18 (1932), 148-159
[11] Kuratowski K., Some remarks on the relation of classical set-valued mappings to the Baire classification, Colloquium Math. XLII (1979), 295-300
[12] Lechicki A., On continuous and measurable multifunctions, Commentations Math., 21 (1979), 141-156
[13] Michel E., Continuous selections I, II, Ann. of Math. 64 (1965), 375-390 , 562-580
[14] Neubrunn T., On quasicontinuity of multifunctions, Math. Slovaca 32, no 2, (1982), 147-154
[15] Ricceri B., Carathéodory's Selections for Multifunctions with Non-Separable Range, Rend. Sem. Mat. Univ. Padova, vol. 67 (1982)185-190
[16] Ricceri B., Selections of multifunctions of two variables, Rocky Mountain Journal of Math., vol. 14 no 3 (1984), 503-517
[17] Sierpiński W., Sur un probleme concernant les ensembles measurables superficiellement, Fund.Math. 1,(1920), 112--115
[18] Shih Shu-Chung, Semi-continuité génerique de multiapplication, Comptes Rendus Acad.Sc.Paris, sér. 1, 293, no 1 (1981), 27-29
[19] Strother W.L., Continuous multi-valued functions, Boll. da Soc. de Mat. de Sao Paolo, 10 (1955), 87-120

MULTIFUNKCJE DWOCH ZMIENNYCH O PÓŁCIĄGŁYCH CIĘCIACH

Streszczenie

Przedmiotem tej pracy są multifunkcje $F: X \times Y \longrightarrow Z$, gdzie X i Y są przestrzeniami metrycznymi, a Z jest przestrzenią metryczną ośrodkową. Główny wynik dotyczy przynależności do górnej klasy 1 multifunkcji F, której wartości są domknięte, a cięcia odpowiednio dolnie i górnie półciągłe. Zilustrowano też patologiczne zachowanie się multifunkcji, których wszystkie cięcia są dolnie półciągłe i wprowadzono koncepcję silnej dolnej półciągłości. W końcowej części pracy podane są warunki dostateczne istnienia selektorów badanych multifunkcji o określonych z góry własnościach.