ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ W BYDGOSZCZY
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ON FUNCTIONS WITH ALMOST EVERYWHERE CONTINUOUS, APPROXIMATELY CONTINUOUS SECTIONS

The present article is devoted to giving a solutions of a problem published by Z.Grande in [5], p.14 and related problems 6 a_3 on p.17, 6c on p.18, 7a on p.19, 12 on p.22 from collection of open problems [6]. All problems under consideration concern real functions defined on the plane, $f: R^2 \to R$, such that all the sections $f_x := f(x, .) : R \to R$, $x \in R$ and $f^y := f(. , y) : R \to R$, $y \in R$ are approximately continuous and/or almost everywhere continuous.

We give some preliminaries about various "fine" typologies to be used in the sequel. A common feature of various kinds of metric density that have hitherto been studied /see [21, 11, 19, 9, 10]/ is that the density of a set E at a point $z \in R^k$ is the limit as n tends to infinity of the mean density of E in C_n, where $\{C_n : n \in N\}$ is any sequence of sets convergent to z, belonging to some family fixed in advance. We recall /cf. [21, 9]/ that a sequence of sets $E_n \subset R^k$, n = 1,2,... is convergent to a point $z_0 \in R^k$ if $z_0 \in \bigcap_{n=1}^{\infty} E_n$ and diam $E_n \to 0$ as n tends to infinity.

The parameter of regularity of a bounded measurable set E of positive diameter is the number $p(E) := \sup_J \{ m_k(E) / m_k(J) \}$ for cubes J containing E, where m_k denotes Lebesgue measure on the Euclidean space R^k. A convergent sequence of measurable sets E_n is regular if there exists a positive constant $A > 0$ such that $p(E_n) \geq A > 0$ for all $n \in N$. Let us mention that an interval in R^k is understood to have sides parallel to axes of coordinates and a cube is an interval with equal non-zero sides.

Let Λ be a family of convergent sequences of measurable sets and for each $z \in R^k$ let $\Lambda(z)$ denote the subfamily consisting of those that converge to z. A measurable set E is said to have a density $d(\Lambda, z, E)$ at z relative to Λ if $\Lambda(z)$ is nonempty and $m_k(C_n \cap E) / m_k(C_n) \to d(\Lambda, z, E)$ as $n \to \infty$ for every sequence $(C_n)_{n=1}^{\infty} \in \Lambda(z)$. It is easy to see that if Λ is the family of all convergent sequences of cubes /resp. non-degenerate intervals/ then $d(\Lambda, z, E) = 1$ if and only if z is an ordinary /resp. strong/ density point of E.

The Lebesgue measure m_k induces on R^k a topology called the Λ-density topology T_d / see [9, 10] /. A set is open in this topology if it is measurable and each of its points is a point of Λ-density one of the set. The Λ- density topology is known to be a completely regular, Hausdorff non-normal topology. Moreover a function $f: R^k \to R$ is ordinarily /resp. strongly/ approximately continuous /cf. [19, 11, 24, 21] / if and only if it is continuous with respect to the Λ- density topology for a suitable family Λ.

Let $T_{a.e.}^{\Lambda}$ be a collection of all subsets $U \subset R^k$ for which $U \in T_d^{\Lambda}$ and $U = G \cup Z$ where G is open /in the Euclidean topology on R^k/ and $m_k(Z) = \emptyset$. It can be proved / see [18] / that $T_{a.e.}^{\Lambda}$ is a

topology on R^k lying between the Euclidean topology T_e and T_d^{Λ}.
We have /cf. [13 , 14] where k=1 / that:

/1/ $T_{a.e.}^{\Lambda} = \{U \in T_d^{\Lambda} : m_k(U) = m_k(\text{Int } U)\}$,

where Int U denotes the Euclidean interior of U. For further generalization using lifting theory see [8].

$T_{a.e.}^{\Lambda}$ is a completely regular Hausdorff non-normal topology on R^k and the class of Λ-approximately continuous functions whose points of T_e - dicontinuity form a set of m_k - measure zero is precisely the collection of $T_{a.e.}^{\Lambda}$ - continuous functions. Moreover $T_{a.e.}^{\Lambda}$ is the coarsest topology T making each such m_k - almost everywhere continuous, Λ- approximately continuous function T - continuous.
Let T_r^{Λ} be the collection of all sets which are the union of some subfamily of the family $F_\sigma(R^k) \cap G_\delta(R^k) \cap T_d^{\Lambda}$. The collection T_r^{Λ} forms a topology which is the coarsest topology making each Λ-
-approximately differentiable function continuous /see [13] for the case k=1 /. We have $T_e \subset T_{a.e.}^{\Lambda} \subset T_r^{\Lambda} \subset T_d$ with proper inclusions. Particular cases of the following auxiliary proposition are already known:

PROPOSITION 1[1]. Let (X, T) be an arbitrary topological space and (I, T_e) the unit interval endowed with the Euclidean topology.
Assume that the function $f: X \times I \to R$ is such that all its Y-sections $f^y : X \to R$, $y \in I$ are T- continuous and all X- sections $f_x : I \to R$ are increasing. If a section $f_u : I \to R$ is continuous at some point $\in I$ /and increasing/ then f is $T \otimes T_e$ - continuous at the point $(u, v) \in X \times I$.

P r o o f: Let $\varepsilon > 0$ be a fixed but arbitrary positive real number. Since f_u is continuous at v, there is a clored ball

$\overline{K}(v,r) \subset K(v,2r) \subset I$ such that:

/2/ $|f(u,v) - f(u,y)| < \varepsilon/2$ for all $y \in \overline{K}(v,r)$.

Since the sections f^{v+r} and f^{v-r} are both continuous at $u \in X$, there are two open neighbourhoods U_1 and U_2 of this point u such that:

/3/ $|f(x, v+r) - f(u, v+r)| < \varepsilon/2$ for all $x \in U_1$, and

/4/ $|f(x, v-r) - f(u, v-r)| < \varepsilon/2$ for all $x \in U_2$.

Observe that $U := U_1 \cap U_2$ is an open neighbourhood of $u \in X$ for which the inequalities /4/ und /3/ are satisfied simultaneously. By virtue of the assumed monotonicity of all sections f_x we have the inequality:

/5/ $f(x, v-r) \leq f(x, y) \leq f(x, v+r)$; $x \in X$.

For $x \in U = U_1 \cap U_2$ and $y \in \overline{K}(v,r)$ we have by the triangle inequality from /3/ and /2/ the subsequent relation:

/6/ $|f(x, v+r) - f(u, v)| \leq |f(x, v+r) - f(u, v+r)| +$

$+ |f(u, v+r) - f(u,v)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

Similarly from /4/ and /2/ we obtain:

/7/ $|f(x, v-r) - f(u,v)| \leq |f(x, v-r) - f(u, v-r)| + |f(u, v-r) -$

$- f(u, v)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

For $(x,y) \in U \times K(v,r)$ the above inequalities yield in the presence of /5/:

/8/ $-\varepsilon < f(x, v-r) - f(u, v) \leq f(x,y) - f(u,v) \leq f(x, v+r) -$

$- f(u, v) < +\varepsilon$ and

/9/ $-\varepsilon < f(u,v) - f(u, v+r) \leq f(u,v) - f(x,y) \leq f(u,v) - f(u, v-r) < \varepsilon$, Combining /8/ and /9/ and using /6/ and /7/ we obtain immediately:

/10/ $|f(x,y) - f(u,v)| \leq \varepsilon$ for $(x,y) \in U \times K(v,r)$.

Since $U \times K(v,r)$ is an open neighbourhood of a point (u,v) in the space $X \times I$ endowed with the product topology $T \otimes T_e$, we infer that /u,v/ is a continuity point of f. The proof is thereby achieved. The subsequent proposition gives an affirmative answer to the question[2/] published by Z. Grande in [5], p.14 .

PROPOSITION 2. Let $f : I^2 \to R$ be a function whose all X-sections f_x and Y - sections f^y are approximately continuous and m_1 - almost everywhere continuous. Then there is a sequence $f_n : R^2 \to R$ of ordinarily approximately continuous and m_2 - almost everywhere continuous functions pointwise convergent to a given function f.

P r o o f: We may assume without any loss of generality that our function f is bounded and positive since in the opposite cas the superposition $h \circ f$ may be considered, where $h: R \to (0,2)$ is an increasing homeomorphism given for example by the formula:

$$/11/ \quad R \ni x \mapsto h(x) := 1 + \operatorname{th} x = 1 + \frac{\exp x - \exp(-x)}{\exp x + \exp(-x)} \in (0,2)$$

Let us introduce the auxiliary function:

$$/12/ \quad I^2 \ni (x,y) \mapsto g(x,y) := \int_0^y f(x,u)\,du \in [0,2)$$

Observe that the x - sections g_x of the function /12/ are continuous and icreasing for all $x \in I$. Next define the set:

$$/13/ \quad A := \{(x,y) \in I^2 : f^y \text{ is continuous at the point } x \in I \text{ and } f_x \text{ is continuous at the point } y \in I\}.$$

All Y- sections $A^y := \{ x \in I : (x,y) \in A \}$ of the set /13/ are of full measure because of the assumption that all f^y are m_x - almost everywhere continuous. Moreover the set A is m_2 - measurable being a intersection of the countable family of sets open in the topology:

/14/ $T_o := \{ U \subset I^2 :$ U is m_2 - measurable and all sections U_x, U^y are open in the Euclidean topology for any $(x,y) \in I^2 \}$.

For topologies of this kind see [15], [7].

The functions continuous on (I^2, T_o) are exactly those separately continuous on the square I^2 and it is well - known that te set of continuity points of a function defined on an arbitrary topological space is a G_δ subset of this space. Then by virtue of famous Fubini theorem we have $m_2(A) = m_2(I^2) = 1$ so that also the sections $A_x = \{ y \in I : (x,y) \in A \} \subset \{ y \in I:$ x is a continuity point of the section $f^y \}$ are of full measure m_y for m_x - almost all points x belonging to I.

Subsequently let us define the set:

/15/ $B := \{ (x, y) \in I^2 : g^y$ is continuous at $x \}$.

In compliance with the theorem 6.1 on page 306 from [22] the section $g^y : I \to (0,2)$, $y \in I$ is continuous at all points $x \in I$ for which m - almost all sections f^y are continuous at x. Therefore $m_x(B^y) = 1$ for $y \in I$ so that we get that the Y - sections of the function g are m_x - almost everywhere continuous. Any point $(x,y) \in I^2$ with the property that the section g_x is continuous at y and increasing and simultaneously the section g^y is continuous at x is by virtue of Proposition 1 a point of joint continuity of g. Applying once again Fubini theorem we conclude that the set of joint continuity points of g is of full plane measure. Thus we have already proved that g is m_2 - almost everywhere continuous. To see that g is ordinarily approximately continuous on the square I^2 firstly let us observe that:

/16/ $$\lim_{v \to x} \text{appr } g(v,y) := \lim_{\substack{v \to x \\ v \in E(x)}} g(v,y) = g(x,y)$$

where $E(x)$ is a subset of I /called sometimes a path leading to x/ such that x is a density point and an accumulation point of $E(x)$ with the property that the restriction $f^y \mid E(x)$ is continuous at x. Such path exists by virtue of the assumed approximate continuity of the sections f^y, $y \in I$.

To prove the equality /16/ ist suffices to verify that[3/]:

$$\frac{m(\{t : |g(x,y) - g(t,y)| < \varepsilon\} \cap [x-h, \quad x+h])}{2h} \xrightarrow[h \quad 0]{} 1$$

But this follows from the fact that f^u is approximately continuous:

$$\lim_{h \to 0} \frac{1}{2h} \int_{x-h}^{x+h} f(v,u) \, dv = f(x,u)$$ and from inclusion

$$\{t: |g(x,y) - g(t,y) < \varepsilon\} \supset \{t: \int_0^y |f(x,u) - f(t,u)| \, du < \varepsilon\}$$

It shows the approximate continuity of all sections g^y.

Specializing the topology T in Proposition 1 to be the density topology T_d on the interval I we deduce that g is $T_d \otimes T_e$ - continuous on the square I^2. But each $T_d \otimes T_e$ - open set is open in the density topology on the square I^2 with respect to the ordinary differentiation base. Thus the function g is ordinarily approximately continuous. Combining this fact with the proved m_2 - almost everywhere continuity of g we obtain that g is $T^{\square}_{a.e.}$ - continuous on I^2. ($\mathcal{A}=\square$ is regular)[4/].

Let h_n, n=1,2,... be a fixed sequence of positive real numbers tending to zero as n tends to infinity. Define the sequence of functions:

$$/17/ \quad I^2 \ni (x,y) \mapsto f_n(x,y) := h_n^{-1} \cdot [g(x,y+h_n) - g(x,y)].$$

All functions /17/ have sectionwise properties the same as the function g and thus are also jointly $T^{\square}_{a.e.}$ - continuous on I^2. All X - sections f_x of our starting function f are approximately continuous and bounded. Hence f_x, $x \in I$ are integrable derivatives and we have the equality:

$$/18/ \quad f(x,y) = \lim_{n \to \infty} f_n(x,y) \quad \text{for all } (x,y) \in I^2 .$$

The proof of Proposition 2 is thereby completed.

COROLLARY 1. Let $f: R^2 \to R$ be a function with $T_{a.e.}$ - continuous all sections f_x and f^y ; $(x,y) \in R^2$. Then f is the pointwise limit of the sequence of $T^{\square}_{a.e.}$ - continuous functions.

P r o o f: Let us decompose the plane R^2 as the countable union of unit squares:

$$/19/ \quad R^2 = \bigcup_{m=-\infty}^{+\infty} \bigcup_{k=-\infty}^{+\infty} [k, k+1] \times [m, m+1] .$$

Applying Proposition 2 to each restriction $f \,|\, [k, k+1] \times [m, m+1]$ and sticking the obtained sequences of functions together we obtain the claimed assertion. /cf. Proposition 6 bellow/.

In connection with Corollary 1 let us recollect the following facts:

a/ Each function $f:R^2 \to R$ with $T_{a.e.}$ - continuous all X- and Y-sections belongs to the Baire class two. Paper [3] contains an example of such function not belonging to the first class of Baire.

b/ There exists a function $f:R^2 \to R$ whose all X - sections and Y-sections are approximately continuous which is totally discontinuous and which is not the pointwise limit of any sequence of m_2 - almost everywhere continuous functions / see [5] /

COROLLARY 2. Let $f: R^2 \to R$ be the same as in Corollary 1. Then f satisfies the following condition /AP_1/ : for each $a < b$ and nonempty sets U and V satisfying

/20/ $U \subset \{(x,y) \in R^2 : f(x,y) < a\}$, $V \subset \{(x,y) \in R^2 : f(x,y) > b\}$

/21/ $U \subset \{(x,y) \in R^2 :$ Cl U has positive ordinary upper density at $(x,y)\}$ and

$V \subset \{(x,y) \in R^2 :$ Cl V has positive ordinary upper density at $(x,y)\}$

it is true that $U \setminus \mathrm{Cl}\ V \neq \emptyset$ or $V \setminus \mathrm{Cl}\ U \neq \emptyset$. The sign Cl stands here for the closure operator in the Euclidean topology on the plane.

P r o o f: This follows easily from Theorem 4.5 on p. 323 from [18], see also [2].

COROLLARY 3. Let $f: R^2 \to R$ be as in the Corollary 1. Then f and -f satisfy the following condition /AP_2/: for each $a < b$ and an arbitrary closed subset $F \subset R^2$ with

/22/ $m_2(F) = m_2\left(F \cap \{(x,y) : f(x,y) < a\}\right) < +\infty$

it is true that the set

/23/ $W := F \cap \{(x,y) : f(x,y) \geq b\}$

possesses the property that

/24/ $(Cl\ S \setminus Cl(W \cap S)) \cap \{(x,y) \in R^2 : f(x,y) \geq b\}$

is a countable intersection of cozero sets in the $T_{a.e.}^{\square}$ - topology on the plane where

/25/ $S := \{(x,y) \in R^2 :$ $Cl\ W$ has positive upper ordinary density at the point $(x,y)\}$.

Let us recall that cozero sets in the a.e. topology are exactly the sets of the form $G \cup Z$ where $G \cup Z$ is open in the ordinary density topology, G is open in the Euclidean topology and Z is an F_σ set of m_2 - measure zero.

P r o o f: This follous from theorems 4.7 and 5.1 on p. 323 in [18].

COROLLARY 4. Let $f: R^2 \to R$ be as in the Corollary 1. Then there is function $f_1 : R^2 \to R$ in the first Baire class and an F_σ set $Z_o \subset R^2$ of m_2 - measure zero such that:

/26/ $\{(x,y) \in R^2 : f(x,y) \neq f_1(x,y)\} \subset Z_o$.

P r o o f: Since all functions /17/ are m_2 - almost everywhere continuous, the inclusion /26/ follows directly from the Theorem 3 of Mauldin [17] generalized in an obvious manner onto the case of functions of two variables see also [16].

The following proposition, based upon results of [18] answers problem 6 b on p. 18 in [6] in the negative:

PROPOSITION 3. There is a function $f: R^k \to R$ being the pointwise limit of m_k - almost eberywhere continuous functions and satisfying condition $/AP_1/$ from Corallary 2 but non expressable as a pointwise limit of $T_{a.e.}^{\square}$ - continuous functions $f_n: R^k \to R$.

P r o o f: Aa in example 6.8 on p. 327 from [18] a function satisfying Grande´s condition $/AP_1/$ but that fails to satisfy condition $/AP_2/$ formulated in Corollary 3 can be exhibited. Bearing in mind that each pointwise limit of $T_{a.e.}^{\square}$ - continuous functions must fulfil the condition $/AP_2/$ in accordance with theorem 4.7 on p. 323 in [18] we obtain the desired thesis.

The subsequent proposition decides the problem 6 a_3 on p.17 from [6] in the positive[5/].

PROPOSITION 4. Let $f: I^2 \to R$ be a function whose all sections f_x and f^y $(x,y) \in I^2$ are approximately continuous. Then f is a pointwise limit of sequence of $T_d \otimes T_d$ - continuous functions, where T_d is the density topology on the interval I.

Pr o o f: As in the proof of Proposition 2 let us assume that f is positive and bounded, Then define the function g by the formula /12/ and observe that it has approximately continuous all sections g^y, $y \in I$ and increasing and continuous all sections g_x, $x \in I$. These properties are inherited by functions f_n defined by the formula /17/. Invoking Proposition 1 for $T = T_d$ we obtain that the functions f_n are $T_d \otimes T_e$ - continuous and thus also $T_d \otimes T_d$ - continuous. That ends the proof.

COROLLARY 5. Each function f: $R^2 \to R$ separatetely approximately continuous /ant therefore continuous with respect to the topology d_{xy} defined in [15] /is a pointwise limit of a sequence of $T_d \otimes T_d$ - continuous functions.

P r o o f: It is exactly the same as the proof of Corollary 1.

Corollaries 1 and 5 may be viewed as a generalization onto the case of a.e. - topology /resp. the density topology/ of the well-known fact that any separately continuous function of two variablew being in the Baire class one is the pointwise limit of the sequence of jointly continuous functions. The r- topology defined in [13] occupies an intermediate place between $T_{a.e.}$ and T_d and also for it we have a similar result:

PROPOSITION 5. Each function f: $R^2 \to R$ whose all sections f_x and f^y, $(x,y) \in R^2$, are r- continuous is a pointwise limit of a sequence of $T_r \otimes T_r$ - continuous functions.

The proof will be ommited, since it is very similar to the given ones. The following extension theorem will be useful in order to solve the problem 12 a on p. 22 in [6]:

PROPOSITION 6. /cf.[23], thm. 3/ The following conditions are equivalent:

/i/ for each Baire 1 function g: $R^k \to R$ there is $T^{\square}_{a.e.}$ - continuous function f: $R^k \to R$ such that the following inclusion holds:

/27/ $\{z \in R^k : f(z) = g(z)\} \supset A$, $A \subset R^k$ (cf.formula (26))

/ii/ the set $A \subset R^k$ fulfils the equality:

/28/ $m_k(Cl\ A) = 0$.

In case k=1 this theorem is proved in [4]. The proof given in [4] does not carry over the multidimensional case. This theorem is obtained in a full generality (Chaika spaces as domaines and Frechet spaces as ranges /in [23] as a consequence of some selection theorem for multifunctions. Proposition 6 itself solves another problem 13 a on p. 23 from [6], but we use it here to decide the question 12 a from [6]. Namely we have:

PROPOSITION 7. There is a function ordinarily approximately continuous and m_2 - a.e. continuous, $f: R^2 \to R$, such that the set:

/29/ $D(f) := \{(x,y) \in R^2 :$ f_x fails to be approximately continuous at y or f^y fails to be approximately continuous at x$\}$ is uncountable.

P r o o f: Let C be a Cantor ternary set in unit interval I. Take $A = I \times C \cup C \times I$ and let $g = R^2 \to R$ be the indicator of the set $C \times C$. The equality /28/ is obviously fulfilled since A is a perfect subset of plane measure zero. Thus the restriction g/A has $T^{\square}_{a.e.}$ - continuous extension $f: R^2 \to R$. For this extension we have $D(f) = C \times C$ so that the set /29/ is uncountable.

In accordance with [20] for each perfect set P of measure zero - $m(P) = 0$, there is an bounded, upper semicontinuous, in the Zahorski class M_4, function $\tilde{f}: R \to R$ such that the set of points of approximate discontinuity of $\tilde{f}$ is exactly the prescribed set P and each point $x \notin P$ is a point of T_e - continuity of $\tilde{f}$. We may use such function $\tilde{f}$ in place of g to obtain the function f in proposition 7 with some additional properties.

Moreover let us recall that the set of approximate continuity points of Baire 1 function $g: R^k \to R$ is residual, Borel and has full measure. Nevertheless a characterisation problem for sets /29/ remains still unresolved.

NOTES:

1/ As it has been remarked by Mirosław Filipczak, the thesis of Proposition 1 holds under significantly weaker assumptions, e.g. if the set $\{y \in I: f^y$ is continuous at $x\}$ is dense in I. A modification of Proposition 1 with still more local character may be also given.

2/ Original formulation /in French/ of the problem is the following La fonction f: $R^2 \to R$ ayant toutes ses sections f_x and f^y continues presque partout et approximativement continues doit-elle etre la limite d'une suite de fonctions continues presque partout?

3/ We omit a piece of routine but tedious verification.

4/ The sign $\square$ means here and in the sequel the family of rectangles $[x-h \ , \ x+h] \times [y-k \ , \ y+k]$ for which a positive constant K exists, such that the ratio h/k fulfils a double inequality: $K^{-1} \leq h/k \leq K$.

That means, $\square$ is an ordinary differentiation basis.

5/ Soit f: $R^2 \to R$ une fonction approximativement continue par rapport a chacune de deux variables. Existe-t-il une suite de fonctions continues par rapport a la topologie produite $d \times d$ convergente en tout point vers f ?

6/ On sait que l'ensemble /29/ peut être dénombrable infini. Peut-il etre indénombrable?

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O FUNKCJACH KTÓRYCH CIĘCIA SĄ APROKSYMATYWNIE CIĄGŁE I PRAWIE WSZĘDZIE CIĄGŁE

Streszczenie

W tym artykule pokazano, że każda funkcja dwóch zmiennych, której wszystkie cięcia poziome i pionowe są ciągłe w topologii a.e. O'Malleya na prostej jest punktową granicą ciągu funkcji ciągłych w topologii a.e. na płaszczyźnie. Rozwiązuje to z nawiązką problem opublikowany przez Z.Grandego w [5] i powtórzony jako problem 6c w [6]. Zastosowana metoda pozwoliła również udzielić odpowiedzi na pytanie 6 a_3 z [6] tzn. pokazać, że funkcja dwóch zmiennych o aproksymatywnie ciągłych wszystkich cięciach jest punktową granicą ciągu funkcji $T_d \otimes T_d$ - ciągłych.
Kolejnym wynikiem tej pracy jest częściowa charakteryzacja zbioru /30/ punktów, w których któreś z cięć aproksymatywnie ciągłej względem zwykłej bazy różniczkowania / i nawet dodatkowo prawie wszędzie ciągłej/ funkcji dwóch zmiennych może być aproksymatywnie nieciągłe. Stwierdzenie 7 pokazując że taki zbiór /30/ może być nieprzeliczalny odpowiada na pytanie 12 a z [6]. Ponadto zauważono, że jeden z przykładów zamieszczonych w [18] stanowi rozwiązanie problemu 6 b z [6], a mianowicie świadczy o tym, że warunek konieczny na to, aby funkcja była granicą ciągu funkcji a.e. - ciągłych sformułowany przez Z.Grandego w [2] nie jest jednocześnie warunkiem wystarczającym. Tematyka tego artykułu może też być rozpatrywana jako badanie własności pewnych topologii na płaszczyźnie skonstruowanych na wzór prac [15] i [7].