ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ
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Mirosław Szatkowski
WSP Bydgoszcz

CONCERNING KRIPKE SEMANTICS FOR INTERMEDIATE PREDICATE LOGICS

In [4] S. Nagai introduced a new type of semantics for intermediate predicate logics, which in [3] H. Ono and S Nagai termed the general Kripke models. In this paper, we distinguish the general Kripke frames from the Kripke models on the general Kripke frames. Besides definig the general Kripke frames, we also define structurally general Kripke frames, and examine certain properties of these two types of semantics.

In I we establish symbols and terminology which are used throughout the paper.

In II we examine the relations between the general Kripke frames and the structurally general Kripke frames. We also examine certain properties of these semantics.

In III we apply the results obtained by the author in [7] to the intermediate predicate logics.

I. Well-known logical and set-theoretical notions and symbols are used in the paper. The symbols: $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\forall$, $\exists$ denote the familiar logical connectives: conjunction, disjunction, implication, negation, universal quantifier and existential quantifier, respectively. The letter N denotes the set of natural numbers (from zero). For each $n \in N$, the symbols: $p^{(n)}, q^{(n)}, r^{(n)}, \ldots$ denote n-ary predicate variables. The symbol AT denotes a set of atomic formulas built in the usual way by means of predicate variables and individual variables from an countably infinite set $\{x, y, z, \ldots\}$, and the symbol FOR denotes a set of formulas built by means of connectives: $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\forall$, $\exists$ and atomic formulas. For any $Q \subset \{\wedge, \vee, \rightarrow, \neg, \forall, \exists\}$ the symbol FOR_Q denotes a set of formulas whose all connectives belong to Q. Small Greek letters α , β , ... will serve as metalinguistic variables running through the set FOR, whereas the symbols: $\bigwedge\!\!\!\wedge$, $\bigvee\!\!\!\vee$, $\Rightarrow$, $\Leftrightarrow$, $\neg\!\!\neg$, $\forall\!\!\!\forall$, $\exists\!\!\!\exists$ stand for conjunction, disjuntion, implication, equivalence, negation, universal quantifier and existential quantifier in the metalanguage. Capital gothic letters $\mathfrak{A}$, $\mathfrak{B}$, . . . denote algebras, and the respective capital Latin letters A, B, . . . the universes of the algebras. The present work will discuss only the so called pseudo-Boolean algebras (cf. [6]),

further called symply algebras. The symbols: $1_{\mathfrak{A}}$, $0_{\mathfrak{A}}$, $\leq_{\mathfrak{A}}$ denote respectively: unit element, zero element, lattice order of the algebra $\mathfrak{A}$, while the symbols: $\wedge_{\mathfrak{A}}$, $\vee_{\mathfrak{A}}$, $\rightarrow_{\mathfrak{A}}$, $\neg_{\mathfrak{A}}$ denote pseudo-Boolean operations of that algebra (cf. [6]). In the above symbols the index $\mathfrak{A}$ will be omitted whenever the possibility of confusion can be excluded.

In this work, by Kripke frame we shall understand any partially ordered set, i.e. a pair $\mathcal{W} = \langle W, \leq \rangle$ such that $W \neq \emptyset$ and $\leq$ is a partial order in W. By a general Kripke frame (g. K. f.) we shall understand a triplet $\langle \mathcal{W}, V, \mathfrak{A} \rangle$ satisfying the following conditions (cf. [3]):

(i) $\mathcal{W}$ is a Kripke frame,

(ii) V is a mapping from W to the power set of some set such that $V(a) \neq \emptyset$ for any $a \in W$ and $V(a) \subseteq V(b)$ if $a \leq b$,

(iii) $\mathfrak{A}$ is a algebra in which there exist $\bigwedge_{t \in T} a_t$ and $\bigvee_{t \in T_1} a_t$

for any T, T_1 such that $\overline{\overline{T}} < \aleph(W, V)$ and $\overline{\overline{T}}_1 < \aleph(V)$,

where $\aleph(W, V)$ denotes the smallest cardinal which is greater than $\overline{\overline{V(a)}}$ and $\overline{\overline{\{b \in W \mid a \leq b\}}}$ for any $a \in W$, and $\aleph(V)$ denotes the smallest cardinal which is is greater than $\overline{\overline{V(a)}}$ for any $a \in W$.

The symbol FOR(V) is used to denote the set of formulas obtained from FOR by adding the individual constansts $\overline{\xi}$ for each element $\xi \in \bigcup\{\overline{\overline{V(a)}} \mid a \in W\}$. By a valuation on the g. K. f. $\langle \mathcal{W}, V, \mathfrak{A} \rangle$

we shall understand every function v from the Cartesian product of the set of all closed formulas of FOR(V) and W into A satisfies the following conditions (cf. [3]):

1. for an n-ary predicate variable $p^{(n)}$, $v(p^{(n)}\ \bar{\xi}_1 \ldots \bar{\xi}_n, a) \leqslant v(p^{(n)}\ \bar{\xi}_1 \ldots \bar{\xi}_n, b)$ if $a \leqslant b$ and $\langle \xi_1, \ldots, \xi_n \rangle \in V(a)^n$,
2. $v(\alpha \wedge \beta, a) = v(\alpha, a) \wedge v(\beta, a)$,
3. $v(\alpha \vee \beta, a) = v(\alpha, a) \vee v(\beta, a)$,
4. $v(\alpha \rightarrow \beta, a) = \bigwedge_{a \leqslant b} (v(\alpha, b) \rightarrow v(\beta, b))$,
5. $v(\neg \alpha, a) = \bigwedge_{a \leqslant b} (\neg v(\alpha, b))$,
6. $v(\forall x \alpha, a) = \bigwedge_{a \leqslant b} \ \bigwedge_{\xi \in V(b)} v(\alpha \bar{\xi}, b)$,
7. $v(\exists x \alpha, a) = \bigvee_{\xi \in V(a)} v(\alpha \bar{\xi}, a)$.

We say that $\alpha \in$ FOR is satisfied by a valuation v on the g. K. f. $\langle \mathcal{W}, V, \mathfrak{U} \rangle$, if $v(\alpha', a) = \mathbb{1}$ for any $a \in W$, where α' is the universal closure of α. If every a valuation v on the g. K. f. $\langle \mathcal{W}, V, \mathfrak{U} \rangle$ satisfies formula α, we say that α is true in the g. K. f. $\langle \mathcal{W}, V, \mathfrak{U} \rangle$. The set of all formulas true in the g. K. f. $\langle \mathcal{W}, V, \mathfrak{U} \rangle$ (the contents of $\langle \mathcal{W}, V, \mathfrak{U} \rangle$) will be denoted by $E(\mathcal{W}, V, \mathfrak{U})$.

By a structurally general Kripke frame (s. g. K. f.) we shall understand any triplet $\langle \mathcal{W}, V, \mathcal{W}_1 \rangle$, which satisfies the following conditions:

(i) $\mathcal{W}$ and $\mathcal{W}_1$ are Kripke frames,

(ii) V is a function from W to the power set of some set such that $V(a) \neq \emptyset$ for any $a \in W$ and $V(a) \subseteq V(b)$ if $a \leqslant_{\mathcal{W}} b$.

By Kripke model on the s. g. K. f. $\langle \mathcal{W}, V, \mathcal{W}_1 \rangle$ we shall understand every function v which takes one of values $\{\mathbb{1}, \Phi\}$ as its value for a triplet $\langle \alpha, a, w \rangle$ of a closed formula $\alpha \in$ FOR(V) and an element $a \in W$ and an element $w \in W_1$, whose values are determined by the following conditions:

1. $v(p^{(n)}\ \overline{\xi}_1 \ldots \overline{\xi}_n, a, w) = 1 \Rightarrow \forall\!\!\!\forall w_1 \geqslant_1 w\ \forall\!\!\!\forall b \geqslant a$
$v(p^{(n)}\ \overline{\xi}_1 \ldots \overline{\xi}_n, b, w_1) = 1$,
2. $v(\alpha \wedge \beta, a, w) = 1 \Leftrightarrow v(\alpha, a, w) = 1 \wedge\!\!\!\wedge v(\beta, a, w) = 1$,
3. $v(\alpha \vee \beta, a, w) = 1 \Leftrightarrow v(\alpha, a, w) = 1 \vee\!\!\!\vee v(\beta, a, w) = 1$,
4. $v(\alpha \rightarrow \beta, a, w) = 1 \Leftrightarrow \forall\!\!\!\forall w_1 \geqslant_1 w\ \forall\!\!\!\forall b \geqslant a\ (v(\alpha, b, w_1) = \Phi \vee\!\!\!\vee v(\beta, b, w_1) = 1)$,
5. $v(\neg\alpha, a, w) = 1 \Leftrightarrow \forall\!\!\!\forall b \geqslant a\ \forall\!\!\!\forall w_1 \geqslant_1 w\ (v(\alpha, b, w_1) = \Phi)$,
6. $v(\forall x\alpha, a, w) = 1 \Leftrightarrow \forall\!\!\!\forall b \geqslant a\ \forall\!\!\!\forall \xi \in V(b)\ (v(\alpha\xi, b, w) = 1)$,
7. $v(\exists x\alpha, a, w) = 1 \Leftrightarrow \exists\!\!\!\exists \xi \in V(a)\ (v(\alpha\overline{\xi}, a, w) = 1)$.

We say that formula $\alpha \in$ FOR is satisfied by a Kripke model v on the s. g. K. f. $\langle W, V, W_1 \rangle$, if $v(\alpha', a, w) = 1$ for every $a \in W$ and for every $w \in W_1$, where α' is the universal closure of α. If every Kripke model on the s. g. K. f. $\langle W, V, W_1 \rangle$ satisfies formula α, we say that α is valid on the s. g. K. f. $\langle W, V, W_1 \rangle$. The set of all formulas valid on the s. g. K. f. $\langle W, V, W_1 \rangle$ (the contents of $\langle W, V, W_1 \rangle$) will be denoted by E (W, V, W_1).

By an easy verification by means induction with respect on the numbre of logical symbols in formula α we have

PROPOSITION. Let v be a Kripke model on the s. g. K. f. $\langle W, V, W_1 \rangle$. Let $a, b \in W$, $a \leqslant b$ and let $w, u \in W_1$, $w \leqslant_1 u$. Then for any $\alpha \in$ FOR, $v(\alpha', a, w) = 1$ implies that $v(\alpha', b, u) = 1$, where α' is the universal closure of α.

The symbol CL is used to denote the set of theorems of classical predicate calculus, whereas the symbol INT is used to denote the set of theorems of intuitionistic predicate calculus. We say that a set of formulas L $\subseteq$ FOR is an intermediate predicate logic if it satisfies the following conditions (cf. [5]):

1. INT $\subseteq$ L $\subseteq$ CL
2. L is closed under the substitution,
3. L is closed under modus ponens,
4. L is closed under the generalization.

II. We discuss relations between the general Kripke frames and the structurally general Kripke frames and we prove of some properties of this frames. For this purpose, we modify the known Fitting method (which allows us to capture certain relations between the Kripke frames and the pseudo-Boolean algebras (cf. [2])).

The following construction allows the correlation with any Kripke frame $\mathcal{W}$ of the respective algebra. Let $\mathcal{W} = \langle W, \leqslant \rangle$ be Kripke frame. A subset H $\subseteq$ W is called a hereditary subset of frame $\mathcal{W}$ if for any a, b $\in$ W it follows from a $\in$ H and a $\leqslant$ b that b $\in$ H. The symbol D($\mathcal{W}$) denotes the class of all hereditary subsets of frame $\mathcal{W}$, whereas the symbol Alg($\mathcal{W}$) denotes an algebra with universum D($\mathcal{W}$) and pseudo-Boolean operations defined as follows: for any $H_1, H_2 \in D(\mathcal{W})$, $H_1 \wedge H_2 = H_1 \cap H_2$, $H_1 \vee H_2 = H_1 \cup H_2$, $H_1 \rightarrow H_2 = \{a \mid a \in W \wedge \forall b \in W\ (a \leqslant b \Rightarrow b \notin H_1 \vee b \in H_2)\}$, $\neg H_1 = H_1 \rightarrow \emptyset$. It is obvious that in Alg($\mathcal{W}$) there exist $\bigwedge_{t \in T} H_t$ and $\bigvee_{t \in T} H_t$, where T, T_1 are sets of any power.

THEOREM 2.1. For any s. g. K. f. $\langle \mathcal{W}, V, \mathcal{W}_1 \rangle$, $E(\mathcal{W}, V, \mathcal{W}_1) = E(\mathcal{W}, V, Alg(\mathcal{W}_1))$.

PROOF. Suppose that $\langle \mathcal{W}, V, \mathcal{W}_1 \rangle$ is a s. g. K. f. and v is Kripke model on the s. g. K. f. $\langle \mathcal{W}, V, \mathcal{W}_1 \rangle$. We define a function v_1 from the Cartesian product of the set of all closed formulas of FOR(V) and W into D($\mathcal{W}_1$) as follows: $v_1(\alpha, a) = \{w \in W_1 \mid v(\alpha, a, w) = 1\}$. It is easy to verify, by induction with respect on the number of logical symbols in α, that v_1 is a voluation on the g. K. f. $\langle \mathcal{W}, V, Alg(\mathcal{W}_1) \rangle$. Now, since the unit element of Alg($\mathcal{W}_1$) is W_1 itself, hence $v_1(\alpha, a) = W_1$ if and only if for all $w \in W_1$ $v(\alpha, a, w) = 1$. The proofs of the remaining steps are easy and will be amitted. Q. E. D.

Let $\langle \mathcal{W}, V, \mathfrak{U} \rangle$ be a g. K. £. A subset F of the universum of algebra $\mathfrak{U}$ is called the filter of algebra $\mathfrak{U}$ if F is non-empty and for any $a, b \in A$, $\{a, b\} \subseteq F$ if and only if $a \wedge b \in F$. A filter F is said to be prime if $F \subset A$ and for any $a, b \in A$, if $a \vee b \in F$, then $\{a, b\} \cap F \neq \emptyset$. We say that a filter F is an $\kappa(W, V)$-filter if it has the following property: for any $\overline{\overline{T}} < \kappa(W, V)$, if $\{a_t \mid t \in T\} \subseteq F$, then $\bigwedge_{t \in T} a_t \in F$. And we say that a filter F is an $\kappa(V)$-filter if it has tho following property: for any $\overline{\overline{T}} < \kappa(V)$, if $\bigvee_{t \in T} a_t \in F$, then $\{a_t \mid t \in T\} \cap F \neq \emptyset$, where $\kappa(V)$ denotes the smallest cardinal which is greater than $\overline{\overline{V(a)}}$ for any $a \in W$. If $\kappa(W, V) \leqslant \aleph_0$, then we identify $\kappa(W, V)$-filters with filters. Similaraly, if $\kappa(V) \leqslant \aleph_0$, then we idnntify $\kappa(V)$-filters with prime filters.

LEMMA 2.1. (i) If $\kappa(W, V) > \aleph_0$, then the $\kappa(W, V)$-filter generated by a non-empty subset A_0 of the universum of algebra $\mathfrak{U}$ is the set of all elements $a \in A$ such that $a \geqslant \bigwedge_{t \in T} a_t$ for some elements $a_t \in A_0$, $t \in T$, and for some $\overline{\overline{T}} < \kappa(W, V)$.
(ii) If $\kappa(W, V) \leqslant \aleph_0$, then the $\kappa(W, V)$-filter generated by a non-empty subset A_0 of the universum of algebra $\mathfrak{U}$ is the set of all elements $a \in A$ such that $a \geqslant a_1 \wedge \ldots \wedge a_n$ for some elements $a_1, \ldots, a_n \in A_0$.

PROOF. By an easy verificatien

This follows easy from Lemma 2.1 that

LEMMA 2.2 (cf. H. Rasiowa and R. Sikorski [6]). Lot a_0 be a olement of universum of algebra $\mathfrak{U}$ and let F be a $\kappa(W, V)$-filter of algebra $\mathfrak{U}$. Then $[F, a_0) = \{a \in A \mid a \geqslant a_0 \wedge c, c \in F\}$ is the $\kappa(W, V)$-filter generated by the set $\{a_0\} \cup F$.

LEMMA 2.3. Let F be a $\kappa(W, V)$-filter in algebra $\mathfrak{U}$ and suppose that $(a \to b) \notin F$. Thon the $\kappa(W, V)$-filter $[F, a)$ generated by F and a does not contain b.

PROOF. Suppose that $b \in [F, a)$. Then by Lemma 2.2 $b \geqslant a \wedge c$ for some $c \in F$. So $c \leqslant a \to b$ and hence $a \to b \in F$. Q. E. D.

LEMMA 2.4. Let F be a proper $\aleph(W, V)$- filter of algebra $\mathfrak{A}$ and suppose $\neg a \notin F$. Then the $\aleph(W, V)$-filter $[F, a)$ generated by F and a is proper.

PROOF. On the strength of Lemma 2.3, since $\neg a = (a \to 0)$. Q. E. D.

LEMMA 2.5. Let a algebra $\mathfrak{A}$ satisfies the following condition: the union set-theoretical of any chain of $\aleph(W, V)$-filters of algebra $\mathfrak{A}$ is a $\aleph(W, V)$-filter in $\mathfrak{A}$. Let F_0 be a $\aleph(W, V)$-filter of algebra $\mathfrak{A}$ and $a \notin F_0$. Then F_0 can be extended to a $\aleph(V)$-filter F such that $a \notin F$.

PROOF. Let us note that if $\aleph(W, V) \leqslant \aleph_0$ then we consider filters and thus each algebra satisfies the condition: the union set-theoretical of any chain of filters is a filter. Thus, if $\aleph(W, V) \leqslant \aleph_0$, then the proof of Lemma is the same as in [2], pp. 25-26. So now we only have to consider the following cases: 1°. $\aleph(W, V) > \aleph_0$ and $\aleph(V) \leqslant \aleph_0$, 2°. $\aleph(V) > \aleph_0$. We omit the proof of Lemma in case 1°, since it is made simple by the proof of Lemma in 2°. To prove 2° let us assume that F_0 is a $\aleph(W, V)$-filter in $\mathfrak{A}$ and $a \notin F_0$. Let K be the collection of all $\aleph(W, V)$-filters in $\mathfrak{A}$ not containing a. The collection K is non-empty, since $F_0 \in K$. Let $\mathcal{C}$ be a chain in $\langle K, \subseteq \rangle$ and $P = \bigcup(X \mid X \in \mathcal{C})$. Since $a \notin X$ for all $X \in \mathcal{C}$, then $a \notin P$ and on the strength of assumptions of Lemma P is a $\aleph(W, V)$-filter in $\mathfrak{A}$. Hence, by Kuratowski-Zorn Lemma, K has a maximal element F. Now, assume that F is not a $\aleph(V)$-filter. Then there exist elements $a_t \in A$, $t \in T$ and $\overline{\overline{T}} < \aleph(V)$, such that $\bigvee_{t \in T} a_t \in F$ and $\{a_t \mid t \in T\} \cap F = \emptyset$. For each $t \in T$, let $F_t = [F, a_t)$ be a $\aleph(W, V)$-filter generated by the set $F \cup \{a_t\}$. Hence $F_t \supset F$ for each $t \in T$. Let us assume that $a \in F_t$ for each $t \in T$. Thus by Lemma 2.2 there exists $c_t \in F$, $t \in T$, so that $a \geqslant c_t \wedge a_t$ for each $t \in T$. Let $c = \bigwedge_{t \in T} c_t$. Since F is a $\aleph(W, V)$-filter then $c \in F$ and $a \geqslant c \wedge a_t$ for each $t \in T$. Hence it follows that $a \geqslant \bigvee_{t \in T} (c \wedge a_t)$. Because $c \wedge \bigvee_{t \in T} a_t = \bigvee_{t \in T} (c \wedge a_t)$ (by [6] p. 135) and $c \wedge \bigvee_{t \in T} a_t \in F$, then $a \in F$ - a contradiction. Therefore there exists $t \in T$ such that $a \notin F_t$. So F is not maximal - a contradiction. Thus F is $\aleph(V)$-filter. Q. E. D.

Let $\mathfrak{U}$ be a algebra. By the symbol $F_p(\mathfrak{U})$ we denote the set of all $\aleph(W, V)$-filters of algebra $\mathfrak{U}$, which are $\aleph(V)$-filters. The symbol $\mathcal{F}_p(\mathfrak{U})$ denotes the Kripke frame $\langle F_p(\mathfrak{U}), \subseteq \rangle$.

THEOREM 2.2. (i) Let $\mathfrak{U}$ be a algebra satisfies the condition: the union set-theoretical of any chain of $\aleph(W, V)$-filters is a $\aleph(W, V)$-filter in $\mathfrak{U}$, Then $E(\mathcal{W}, V, \mathfrak{U}) \subseteq E(\mathcal{W}, V, \mathcal{F}_p(\mathfrak{U}))$.

(ii) For any finite algebra $\mathfrak{U}$, $E(\mathcal{W}, V, \mathfrak{U}) = E(\mathcal{W}, V, \mathcal{F}_p(\mathfrak{U}))$.

PROOF. To prove (i) let us assume that $\langle \mathcal{W}, V, \mathfrak{U} \rangle$ is a g. K. f. and v is a voluation on the g. K. f. $\langle \mathcal{W}, V, \mathfrak{U} \rangle$. We define a function v_1 from Cartesian product of the set of all closed formulas of FOR(V), of W and $F_p(\mathfrak{U})$ into $\{ 1\!\!1 , \Phi \}$ as follows: $v_1(\alpha, a, w) = 1\!\!1$ if $v(\alpha, a) \in w$. In order to show that thus-defined function v_1 is Kripke model on the s. g. K. f. $\langle \mathcal{W}, V, \mathcal{F}_p(\mathfrak{U}) \rangle$ we verify only the case 4 and 6, because the verification of the remaining cases is either easy, or similar to 4 or 6. Let $v_1(\alpha \to \beta, a, w) = 1\!\!1$. Then $v(\alpha \to \beta, a) \in w$. So $\bigwedge_{a \leqslant b} (v(\alpha, b) \to v(\beta, b)) \in w$ and hence $v(\alpha, b) \to v(\beta, b) \in w$ for any $b \geqslant a$. If $w_1 \supseteq w$ then is clear that for any $b \geqslant a$, $v(\alpha, b) \to v(\beta, b) \in w_1$. Therefore on basis of the definition of filter, if $v(\alpha, b) \in w_1$, then $v(\beta, b) \in w_1$, and consequently for any $b \geqslant a$ and for any $w_1 \supseteq w$ $v_1(\alpha, b, w_1) = \Phi$ or $v_1(\beta, b, w_1) = 1\!\!1$. Now, consider the case when $v_1(\alpha \to \beta, a, w) = \Phi$. From this assumption it follows that $v(\alpha \to \beta, a) \notin w$ and consequently $\bigwedge_{a \leqslant b} (v(\alpha, b) \to v(\beta, b)) \notin w$. Therefore there exists $b_o \geqslant a$ such that $v(\alpha, b_o) \to v(\beta, b_o) \notin w$. Thus on the strength of Lemma 2.3 we can assume that there exists a $\aleph(W, V)$-filter w_o in algebra $\mathfrak{U}$ such that $w_o \supseteq w$, $v(\alpha, b_o)$

$\in w_o$ and $v(\beta, b_o) \notin w_o$. By applying to the latter Lemma 2.5 we obtain that w_o can be extended to a $\mathcal{K}(V)$-filter w_1 such that $v(\beta, b_o) \notin w_1$. So there exist $b_o \geqslant a$ and $w_1 \supseteq w$ such that $v_1(\alpha, b_o, w_1) = 1\!\!1$ and $v_1(\beta, b_o, w_1) = \mathbb{O}$. In the case 6 the following equivalences hold: $v_1(\forall x\, \alpha, a, w) = 1\!\!1$ iff $v(\forall x\, \alpha, a) \in w$ iff $\bigwedge_{b \geqslant a} \bigwedge_{\xi \in V(b)} v(\alpha \overline{\xi}, a) \in w$ iff $\forall b \geqslant a\, \forall \xi \in V(b)$ $v(\alpha \overline{\xi}, a) \in w$ iff $\forall b \geqslant a\, \forall \xi \in V(b)\ v_1(\alpha \overline{\xi}, a, w) = 1\!\!1$.
To end the proof of (i) it suffices to note that if $v(\alpha, a) = 1\!\!1$ then $v(\alpha, a) \in w$ for all $w \in F_p(\mathfrak{U})$. Hence $v_1(\alpha, a, w) = 1\!\!1$. Conversely, let $v(\alpha, a) \neq 1\!\!1$. Then by Lemma 2.5 we can extend $\{1\!\!1\}$ to a $\mathcal{K}(V)$-filter w such that $v(\alpha, a) \notin w$, and consequently $v_1(\alpha, a, w) = \Phi$.

To prove (ii) observe that by Theorem 2.1, $E(\mathcal{W}, V, \mathcal{F}_p(\mathfrak{U})) = E(\mathcal{W}, V; Alg(\mathcal{F}_p(\mathfrak{U})))$. Since $\mathfrak{U}$ is a finite algebra, therefore $\mathfrak{U}$ is isomorphic with the algebra $Alg(F_p(\mathfrak{U}))$ - the function f: $A \mapsto D(\mathcal{F}_p(\mathfrak{U}))$ such that $f(a) = \{ F \mid a \in F \wedge F \in F_p(\mathfrak{U})\}$, is an isomorphism. Thus we conclude the proof of Theorem. Q. E. D.

Let $\mathcal{W}_1 = \langle W_1, \leqslant_1 \rangle$ and $\mathcal{W}_2 = \langle W_2, \leqslant_2 \rangle$ be Kripke frames; we assume that $W_1 \cap W_2 = \emptyset$. The product of frames $\mathcal{W}_1 \times \mathcal{W}_2$ is defined as a pair $\langle W_1 \cup W_2, \leqslant_{\mathcal{W}_1 \times \mathcal{W}_2} \rangle$, where $\leqslant_{\mathcal{W}_1 \times \mathcal{W}_2} = \leqslant_1 \cup \leqslant_2$. It can be easily noticed that the condition $W_1 \cap W_2 = \emptyset$ is not essential - the product of two Kripke frames can be obtained by using their isomorphic copies. In the case when we consider of the set of Kripke frames $\{\mathcal{W}_i \mid i \in I\}$ we shall write $\times(\mathcal{W}_i) \mid i \in I)$ or $\underset{i \in I}{\times} \mathcal{W}_i$ instead of $\langle \bigcup\{W_i \mid i \in I\},$ $\leqslant_{\underset{i \in I}{\times} \mathcal{W}_i} \rangle$, where $\leqslant_{\underset{i \in I}{\times} \mathcal{W}_i} = \bigcup\{\leqslant_i \mid i \in I$.
An important property of structurally general Kripke frames is. exressed in the following:

THEOREM 2.3. For any set of s. g. K. f.'s $\{\langle \mathcal{W}_i, v_i, W_i^o \rangle$

$\mid i \in I\}$ there is a s. g. K. f. $< \mathcal{W}, V, \mathcal{W}^{\circ} >$ such that $E(\mathcal{W}, V, \mathcal{W}^{\circ}) = \bigcap_{i \in I} E(\mathcal{W}_i, V_i, \mathcal{W}_i^{\circ})$.

PROOF. Suppose that is given of the set of s. g. K. f.'s $\{< \mathcal{W}_i, V_i, \mathcal{W}_i^{\circ} > \mid i \in I\}$. Let us consider the s. g. K. f. $< \mathcal{W}, V, \mathcal{W}^{\circ} >$, where $\mathcal{W} = \underset{i \in I}{\times} \mathcal{W}_i$, $\mathcal{W}^{\circ} = \underset{i \in I}{\times} \mathcal{W}_i^{\circ}$ and V is the function from $\bigcup\{W_i \mid i \in I\}$ such that for any a $\in \bigcup\{W_i \mid i \in I\}$, $V(a) = V_i(a)$ if $a \in W_i$. We omit the proof of the inclusion $E(\mathcal{W}, V, \mathcal{W}^{\circ}) \subseteq \bigcap_{i \in I} E(\mathcal{W}_i, V_i, \mathcal{W}_i^{\circ})$ since it is completly direct. To prove the converse inclusion let us assume that $\alpha \notin E(\mathcal{W}, V, \mathcal{W}^{\circ})$. Thus there exist Kripke model v on the s. g. K. f. $< \mathcal{W}, V, \mathcal{W}^{\circ} >$, $a_o \in W$ and $w_o \in W^{\circ}$ such that $v(\alpha', a_o, w_o) = \Phi$. Since $W = \bigcup\{W_i \mid i \in I\}$ and $W^{\circ} = \bigcup\{W_i^{\circ} \mid i \in I\}$, then here exist $i, j \in I$ such that $a_o \in W_i$ and $w_o \in W_j^{\circ}$. On the strength of the construction of the s. g. K. f. $< \mathcal{W}, V, \mathcal{W}^{\circ} >$ it is obvious that we can assume that $i = j$. Consider Kripke model v_1 on the s. g. K. f. $< \mathcal{W}_i, V_i, \mathcal{W}_i^{\circ} >$ such that $v_1(\beta', a, w) = v(\beta', a, w)$ for each $\beta \in AT$, for each $a \in W_i$ and for each $w \in W_i^{\circ}$. The reader can easily check that so defined of the model v_1 can be extended that: for any $\beta \in FOR$, for any $a \in W_i$ and for any $w \in W_i^{\circ}$, $v_1(\beta', a, w) = v(\beta', a, w)$. Hence, $v_1(\alpha', a_o, w_o) = \Phi$ and so $\alpha \notin \bigcap_{i \in I} E(\mathcal{W}_i, V_i, \mathcal{W}_i^{\circ})$. Q. E. D.

THEOREM 2.4. For any s. g. K. f. $< \mathcal{W}, V, \mathcal{W}^{\circ} >$ there exists a set of s. g. K.'s $\{< \mathcal{W}_i, V_i, \mathcal{W}_i^{\circ} > \mid i \in I\}$ such that

(i) $E(\mathcal{W}, V, \mathcal{W}^{\circ}) = \bigcap_{i \in I} E(\mathcal{W}_i, V_i, \mathcal{W}_i^{\circ})$.

(ii) For any $i \in I$, $\mathcal{W}_i$ has the least element,

(iii) For any $i \in I$, $\mathcal{W}_i^{\circ}$ has the least element.

PROOF. Let us assume that $< \mathcal{W}, V, \mathcal{W}^{\circ} >$ is a s. g. K. f. For each $a \in W$, let $W_a = \{b \mid a \leq b\}$ and let V_a be the restriction of V to W_a. Similarly, for each $w \in W^{\circ}$, let $W_w^{\circ} = \{u \mid w \leq u\}$.

Now let us consider the set of s. g. K. f. 's $\{<\mathcal{W}_a, V_a, \mathcal{W}^{\bullet}_w>|$ $a \in W, w \in W^{\bullet}\}$. It is clear that in order to prove (i) - (iii) it suffices to show: $E(\mathcal{W}, V, \mathcal{W}^{\circ}) = \bigcap\{E(\mathcal{W}_a, V_a, \mathcal{W}^{\circ}_w) \mid a \in W,$ $w \in W^{\bullet}\}$. If $\alpha \in E(\mathcal{W}, V, \mathcal{W}^{\bullet})$, then for every Kripke model v on the s. g. K. f. $<\mathcal{W}, V, \mathcal{W}^{\circ}>$, for every $a \in W$ and for every $w \in W^{\bullet}$, $v(\alpha', a, w) = 1$. This gives that for every $a \in W$ and for every $w \in W^{\bullet}$, $\alpha \in E(\mathcal{W}_a, V_a, \mathcal{W}^{\circ}_w)$, hence $\alpha \in \bigcap\{E(\mathcal{W}_a, V_a, \mathcal{W}^{\circ}_w) \mid a \in W, w \in W^{\circ}\}$. Thus there exist Kripke model v on the s. g. K. f. $<\mathcal{W}, V, \mathcal{W}^{\bullet}>$, $a_o \in W$ and $w_o \in W^{\bullet}$ such that $v(\alpha', a_o, w_o) = \Phi$. This yields - similarly as in the proof of the Theorem 2.3 - that $\alpha \notin E(\mathcal{W}_{a_o}, V_{a_o}, \mathcal{W}_{w_o})$ and therefore $\alpha \notin \bigcap\{E(\mathcal{W}_a, V_a, \mathcal{W}^{\circ}_w) \mid a \in W, w \in W^{\circ}\}$. Q. E. D.

Following H. Ono [9] we say that the function $f: W \mapsto W^{\bullet}$ is a embedding of $\mathcal{W} = <W, \leqslant>$ into $\mathcal{W}^{\circ} = <W, \leqslant^{\circ}>$ if and only if it satisfies the following conditions:

(i) $f(W) = W^{\bullet}$,

(ii) $\forall a, b \in W\ (a \leqslant b \Rightarrow f(a) \leqslant^{\circ} f(b))$,

(iii) $\forall a \in W\ \forall w \in W^{\bullet}\ (f(a) \leqslant^{\circ} w \Rightarrow \exists c \in W(a \leqslant c \wedge w = f(c)))$.

If there is a embedding of $\mathcal{W}$ into $\mathcal{W}^{\circ}$, we say that $\mathcal{W}$ is embeddable in $\mathcal{W}^{\circ}$.

We say that s. g. K. f. $<\mathcal{W}, V, \mathcal{W}_1>$ is embeddable into s. g. K. f. $<\mathcal{W}^{\bullet}, V^{\circ}, \mathcal{W}^{\circ}_1>$ if the following conditions hold:

(i) there exists a embedding f of $\mathcal{W}$ into $\mathcal{W}^{\bullet}$,

(ii) there exists a function g from $\bigcup\{V(a) \mid a \in W\}$ to $\bigcup\{V^{\bullet}(b) \mid b \in W^{\bullet}\}$ such that $g(V(a)) = V^{\bullet}(f(a))$ for each $a \in W$,

(iii) there exists a embedding h of $\mathcal{W}_1$ into $\mathcal{W}^{\bullet}_1$.

THEOREM 2.5. Suppose that $<\mathcal{W}, V, \mathcal{W}_1>$ and $<\mathcal{W}^{\circ}, V^{\circ}, \mathcal{W}^{\circ}_1>$ are structurally general Kripke frames. If $<\mathcal{W}, V, \mathcal{W}_1>$ is embeddable into $<\mathcal{W}^{\bullet}, V^{\circ}, \mathcal{W}^{\circ}_1>$, then $E(\mathcal{W}, V, \mathcal{W}_1) \subseteq E(\mathcal{W}^{\circ}, V^{\bullet}, \mathcal{W}^{\circ}_1)$.

PROOF. Let $\alpha_0 \notin E(\mathcal{W}^o, V^o\ \mathcal{W}_1^o)$. Then there is Kripke model v on the s. g. K. f. $\langle \mathcal{W}^o, V^o, \mathcal{W}_1^o \rangle$, $a_0 \in W^o$ and $w_0 \in W_1^o$ such that $v(\alpha_0, a_0, w_0) = \Phi$. Now, we define Kripke model v_1 on the s. g. K. f. $\langle \mathcal{W}, V, \mathcal{W}_1 \rangle$ as follows: $v_1(\alpha, a, w) = v(\alpha, f(a), h(w))$ for any $\alpha \in AT$, for any $a \in W$ and for any $w \in W_1$, where f is a embedding of $\mathcal{W}$ into $\mathcal{W}^o$ and h is a embedding of $\mathcal{W}_1$ into $\mathcal{W}_1^o$. By induction with respect to the number of logical symbols in formula α, we can show that v_1 is really Kripke model on the s. g. K. f. $\langle \mathcal{W}, V, \mathcal{W}_1 \rangle$ and that for any $\alpha \in FOR$, $v_1(\alpha, a, w) = v(\alpha, f(a), h(a))$. Since $f(W) = W^o$ and $h(W_1) = W_1^o$, then there exist $b \in W$ and $u \in W_1$ such that $f(b) = a_0$, $h(u) = w_0$. Hence, $v_1(\alpha'_0, b, u) = v(\alpha'_0, a_0, w) = \Phi$. Q. E. D.

We say that an algebra $\mathcal{U}$ is strongly compact if and only if there exists the greatest element in set $A - \{1\!\!1\}$. Such an element (if exists) will be denoted by $*_{\mathcal{U}}$. The symbol $\mathcal{U}/F$ denotes the quotient algebra obtained by means of the relation congruence determined by the filter F of algebra $\mathcal{U}$.

LEMMA 2.6. Let $\langle \mathcal{W}, V, \mathcal{U} \rangle$ be a g. K. f. Let $\mathcal{U}$ be a algebra satisfies the condition: the union set-theoretical of any chain of $\kappa(W, V)$-filters is $\kappa(W, V)$-filter in $\mathcal{U}$. Let $a \in A$ and $a \neq 1\!\!1$. Then there exists $\kappa(W, V)$-filter F such that $\mathcal{U}/F$ is a strongly compact algebra and $*_{\mathcal{U}/F} = [a]F$.

PROOF. On the strength of Lemma 2.5 we can extend $\{1\!\!1\}$ to $\kappa(V)$-filter F such that $a \notin F$. Now, we will prove that $\mathcal{U}/F$ is strongly compact and $*_{\mathcal{U}/F} = [a]F$. By a direct argument we get that $1\!\!1_{\mathcal{U}/F} = [1\!\!1]F = F$ It is clear that $[a]F \neq F$, since $a \notin F$. To prove that $[a]F$ is the greatest element in $A/F - \{F\}$ let us suppose that $[b]F \neq F$. Hence $b \notin F$, and so by the proof of Lemma 2.5 $a \in [F \cup \{b\})$, where $[F \cup \{b\})$ is a $\kappa(W, V)$-filter generated by the set $F \cup \{b\}$. Therefore, on the basis of Lemma 2.2 $a \geqslant c \wedge b$ for some $c \in F$. Thus $b \rightarrow a \geqslant c$ which results in $b \rightarrow a \in F$ and consequently $[b]F \leq_{\mathcal{U}/F} [a]F$.
Q. E. D.

THEOREM 2.6. If $\mathfrak{U}$ is a nongenerate algebra then for any g. K. f. $< \mathcal{W}, V, \mathfrak{U} >$ there exists a set of g. K. f. 's $\{ < \mathcal{W}_i, V_i, \mathfrak{U}_i > \mid i \in I \}$ such that

(i) $E(\mathcal{W}, V, \mathfrak{U}) = \bigcap_{i \in I} E(\mathcal{W}_i, V_i, \mathfrak{U}_i)$.
(ii) For any $i \in I$, $\mathcal{W}_i$ has the least element,
(iii) For any $i \in I$, $\mathfrak{U}_i$ is strongly compact algebra.

PROOF. Suppose that $< \mathcal{W}, V, \mathfrak{U} >$ is a g. K. f. For each $a \in W$, let $W_a = \{ b \mid a \leqslant b \}$ and let V_a be he restriction of V to W_a. Let the symbol SC denote a set of all strongly compact algebras, which are of form $\mathfrak{U}/F$. Let us consider the set of g. K. f. 's $\{ < \mathcal{W}_a, V_a, \mathcal{B} > \mid a \in W, \ \mathcal{B} \in SC \}$. It is clear tha in order to prive (i) - (iii) it suffices to show: $E(\mathcal{W}, V, \mathfrak{U}) = \bigcap E(\mathcal{W}_a, V_a, \mathcal{B}) \mid a \in W, \ \mathcal{B} \in SC \}$. The inclusion from the left to right is obvious. To prove the converse inclusion let us assume that $\alpha \notin E(\mathcal{W}, V, \mathfrak{U})$. Thus here exist a valuation v on the g. K. f. $< \mathcal{W}, V, \mathfrak{U} >$ and $a \in W$ such that $v(\alpha', a) \neq 1\!\!1$. Hence, on the strength of Lemma 2.6 there must exist a strongly compact algebra $\mathfrak{U}/F$ such that $*_{\mathfrak{U}/F} = [v(\alpha', a)]F$. Consider a valuation v_1 on the g. K. f. $< \mathcal{W}_a, V_a, \mathfrak{U}/F >$ such that $v_1(\beta, b) = [v(\beta, b)]F$ such $v_1(\beta, b) = [v(\beta, b)]F$ for each $\beta \in AT$ occurring in and for each $b \in W_a$. We omit the proof that $v_1(\alpha', a) = [v(\alpha', a)]F = *_{\mathfrak{U}/F}$, since it is quite simple, therefore, $\alpha \notin E(\mathcal{W}_a, V_a, \mathfrak{U}/F)$. Q. E. D.

THEOREM 2.7. (i) $E(\mathcal{W}, V, \mathfrak{U})$ is an intermediate predicate logic if and only if there is an element $a \in W$ such that $\overline{\overline{V(a)}} \geqslant \aleph_0$.

(ii) $E(\mathcal{W}, V, \mathcal{W}_1)$ is an intermediate predicate logic if and only if there is an element $a \in W$ such that $\overline{\overline{V(a)}} \geqslant \aleph_0$.

PROOF. To prove (i) let us observe that by Theorem 3.6 of [5] p. 629, $E(\mathcal{W}, V, \mathcal{R})$ is an intermediate predicate logic if and only if there is $a \in W$ such that $V(a) \geqslant \aleph_0$, where the letter denotes the two element algebra. Since $\mathcal{R}$ is an subalgebra of any nondegenera e algebra, then is clear that (i) holds.

To prove (ii) let us observe that by Theorem 2.2(ii) $E(\mathcal{W}, v, \mathcal{R}) = E(\mathcal{W}, v, \mathcal{F}_p(\mathcal{R}))$. Now, it is obvious that $\langle \mathcal{W}, v, \mathcal{W}_1 \rangle$ is embeddable into $\langle \mathcal{W}, v, \mathcal{F}_p(\mathcal{R}) \rangle$ for any Kripke frame $\mathcal{W}_1$. Consequently, by Theoreme 2.5, $E(\mathcal{W}, v, \mathcal{W}_1) \subseteq E(\mathcal{W}, v, \mathcal{F}_p(\mathcal{R}))$, which gives finally that (ii) holds. Q. E. D.

III. This section is a continuation of [7]. We give some criterion for the inclusion relation between the non-disjunctive contents of two structurally general Kripke frames.

Let $\mathcal{W} = \langle W, \leqslant \rangle$ be a Kripke frame. For any $a, b \in W$, we say that b is successor of a (symb. $a < b$) if $a \leqslant b$ and $a \neq b$. We say that b is a direct successor of a (symb. $a \prec b$) if $a < b$ and there does not exist $c \in W$ such that $a < c < b$. A Kripke frame $\mathcal{W} = \langle W, \leqslant \rangle$ is strongly atomic (cf. [1]) if and only if for any $a, b \in W$, if $a < b$, then there exists $c \in W$ such that $a \prec c \leqslant b$. A Kripke frame satisfies the increasing sequences condition (cf. [1]) if and only if there does not exist an infinite increasing sequence of elements of that frame. It can easily be seen that a frame satisfies the increasing sequences condition if and only if each of its non-empty subsets has a maximal element.

LEMMA 3.1. Let $\mathcal{W}_1 = \langle W_1, \leqslant_1 \rangle$ be a strongly atomic Kripke frame. Suppose that $X \subseteq W_1$ contains no maximal element of the frame $\mathcal{W}_1$, and let v be a model on the s.g.K.f. $\langle \mathcal{W}, v, \mathcal{W}_1 \rangle$ such that the conditions $\forall \alpha \in AT\ \forall a \in W\ \forall w \in X: \quad v(\alpha, a, w) = \min\{v(\alpha, a, u) \mid w \prec_1 u\}$ is satisfied. Then for any formula $\alpha \in FOR_{\{\wedge, \to, \neg, \forall, \exists\}}$ the condition $\forall a \in W\ \forall w \in X: v(\alpha, a, w) = \min\{v(\alpha, a, u) \mid w \prec_1 u\}$ holds.

PROOF. By iduction with respect on the number of logical symbols in α. For $\alpha \in AT$ the Lemma holds by assumption. Let us suppose for induction that the results is true for all $\beta, \gamma \in FOR_{\{\wedge, \to, \neg, \forall, \exists\}}$ containing less that r logical..symbols and for any $w \in X$ and for any $a \in W$: $v(\beta, a, w) = \min\{v(\beta, a, u) \mid w \prec_1 u\}$ and $v(\gamma, a, w) = \min\{v(\gamma, a, u) \mid w \prec_1 u\}$. Suppose also that α is a formula containing r logical symbols. Let $\alpha = \beta \wedge \gamma$, $a \in W$ and $w \in X$. Then $v(\alpha, a, w) = v(\beta \wedge \gamma, a, w) = \min\{v(\beta, a, w), v(\gamma, a, w)\} = \min\{\min\{v(\beta, a, u) \mid w \prec_1 u\}, \min\{v(\gamma, a, u) \mid w \prec_1 u\}\} = \min\{v(\beta \wedge \gamma, a, u) \mid w \prec_1 u\} = \min\{v(\alpha, a, u) \mid w \prec_1 u\}$. Let us now assume that $\alpha = \beta \to \gamma$, $a \in W$ and $w \in X$. If $v(\beta \to \gamma, a, w) = \mathbb{1}$, then for any $u \geqslant_1 w$ $v(\beta \to \gamma, a, u) = \mathbb{1}$. Since w is not a maximal element in $\mathcal{W}_1$ and $\mathcal{W}_1$ is strongly atomic, then $\{u \mid w \prec_1 u\} \neq \emptyset$, therefore $\min\{v(\beta \to \gamma, a, u) \mid w \prec_1 u\} = \mathbb{1}$. Now, consider the case when $v(\beta \to \gamma, a, w) = \Phi$. From this assumption it follows that for some $z \geqslant_1 w$ and for some $b \geqslant a$: $v(\beta, b, z) = \mathbb{1}$ and $v(\gamma, b, z) = \Phi$. If $z = w$, then according to the induction hypothesis there exists u_1, $w \prec_1 u_1$ and $v(\beta, b, u_1) = \mathbb{1}$ and $v(\gamma, b, u_1) = \Phi$. Thus $v(\beta \to \gamma, b, u_1) = \Phi$ and from here $v(\beta \to \gamma, a, u_1) = \Phi$ which results in $\min\{v(\beta \to \gamma, a, u) \mid w \prec_1 u\} = \Phi$. Let us now suppose that $z \neq w$. Then $w <_1 z$, so by the assumption that $\mathcal{W}_1$ is strongly atomic, there exists u_o such that $w \prec_1 u_o \leqslant z$. Since $u_o \leqslant_1 z$, then $v(\beta \to \gamma, b, u_o) = \Phi$ and $v(\beta \to \gamma, a, u_o) = \Phi$, while since $w \prec_1 u_o$, then $\min\{v(\beta \to \gamma, a, u) \mid w \prec_1 u\} = \Phi$. For $\alpha = \neg \beta$, the reasoning is similiar. Let us suppose that $\alpha = \forall x \beta$, $a \in W$ and $w \in X$. Evidently $\{u \mid w \prec_1 u\} \neq \emptyset$. If $v(\forall x \beta, a, w) = \mathbb{1}$, then for any $b \geqslant a$ and for any $\xi \in V(b)$, $v(\beta \bar{\xi}, b, w) = \mathbb{1}$. Hence by the induction hypothesis for any $b \geqslant a$ and for any $\xi \in V(b)$, $\min\{v(\beta \bar{\xi}, b, u) \mid w \prec_1 u\} = \mathbb{1}$, and so $\min\{v(\forall x \beta, a, u) \mid w \prec_1 u\} = \mathbb{1}$. In the opposite way, if $v(\forall x \beta, a, w) = \phi$, then there is $b \geqslant a$ and $\xi \in V(b)$ such that $v(\beta \bar{\xi}, b, w) = \phi$ and hence $v(\beta \bar{\xi}, a, w) = \phi$. Then on the strength of the induction hipothesis $\min\{v(\beta \bar{\xi}, a, u) \mid w \prec_1 u\} = \phi$, so $\min\{v(\forall x \beta, a, u) \mid w \prec_1 u\} = \phi$. Finally suppose that $\alpha = \exists x \beta$, $a \in W$ and $w \in X$. Similarly as above $\{u \mid w \prec_1 u\} \neq \phi$. If $v(\exists x \beta, a, w) = \mathbb{1}$, then there is some $\xi \in V(a)$ such that $v(\beta \bar{\xi}, a, w) = \mathbb{1}$. Therefore by the induction hypothesis $\min\{v(\beta \bar{\xi}, a, u) \mid w \prec_1 u\} = \mathbb{1}$ and hence

$\min\{v(\exists x\beta, a, u) \mid w \prec_1 u\} = 1$. If $v(\exists x\beta, a, w) = \phi$, then for any $\xi \in V(a)$, $v(\beta\bar{\xi}, a, w) = \phi$. So by the induction hipothesis for any $\xi \in V(a)$, $\min\{v(\beta\bar{\xi}, a, u) \mid w \prec_1 u\} = \phi$, therefore $\min\{v(\exists x\beta, a, u) \mid w \prec_1 u\} = \phi$. Q.E.D.

LEMMA 3.2. Let $\mathcal{W}_1 = \langle W_1, \leqslant_1 \rangle$ be a strongly atomic Kripke frame satisfying the increasing sequences condition. Let $W_o \subseteq W_1$ be a subset containing all maximal elements of the frame $\mathcal{W}_1$. Then every model v_o on the s. g. K. f. $\langle \mathcal{W}, V, \mathcal{W}_o \rangle$ can be extended to a model v on the s. g. K. f. $\langle \mathcal{W}, V, \mathcal{W}_1 \rangle$ so that for any formula $\alpha \in FOR_{\{\wedge, \rightarrow, \neg, \forall, \exists\}}$ for any $a \in W$ and or any $w \in W_o$, $v(\alpha, a, w) = v_o(\alpha, a, w)$.

PROOF. Let the assumptions of the Lemma be satisfied and let v_o be a model on the s. g. K. f. $\langle \mathcal{W}, V, \mathcal{W}_o \rangle$. For any $\alpha \in AT$ we put $v(\alpha, a, w) = v_o(\alpha, a, w)$ if $w \in W_o$, and $v(\alpha, a, w) = \min\{v(\alpha, a, u) \mid w \prec_1 u\}$ if $w \in W_1 - W_o$. Such a definition is correct because, if $w \in W_1 - W_o$, then w is not the maximal element of the frame $\mathcal{W}_1$, thus on the strength of strong atomicity of the frame $\mathcal{W}_1$ the set $\{u \mid w \prec_1 u\}$ cannot be empty. We shall first prove that the function v has been, in the above way, defined for every triplet from the set $AT \times W \times W_1$. Let us suppose that $Y \subseteq W_1$ is such that $u \in Y$ if and only if the function v is defined for all triplet from the set $AT \times W \times \{u\}$. It is clear that $W_o \subseteq Y$ and for any $w \in W_1 - W_o$ if $\{u \mid w \prec_1 u\}$, then $w \in Y$. Let us assume to the contrary, i.e. that $Y \subset W_1$. Then $W_1 - Y \neq \phi$, thus there exists a maximal element in the set $W_1 - Y$ because the increasing sequences condition is satisfied on the strength of the assumption. Let w_o be a maximal element in the set $W_1 - Y$. Then $w_o \in W_1 - W_o$ and $\{u \mid w_o \prec_1 u\} \nsubseteq Y$ because $w_o \notin Y$. Therefore there exists u_o such that $w_o \prec_1 u_o$, $u_o \notin Y$, thus w_o, contrary to the definition, is not maximal in set $W_1 - Y$. We have demonstrated that the function v is defined for every triplet from the set $AT \times W \times W_1$. It is obivious that v must be a model on the s. g. K. f. $\langle \mathcal{W}, V, \mathcal{W}_o \rangle$.

Now , it remains to prove that $\forall\alpha \in FOR_{\{\wedge, \rightarrow, \neg, \forall, \exists\}}$ $\forall a \in W \; \forall w \in W_0 : \; v(\alpha, a, w) = v_0(\alpha, a, w)$.

Let us assume to the contrary . Let $\alpha_0 \; FOR_{\{\wedge, \rightarrow, \neg, \forall, \exists\}}$ will be a formula such that for some $w \in W_0$ and some $a_0 \in W$, $v(\alpha_0, a_0, w) \neq v_0(\alpha_0, a_0, w)$. We known that a formula α_0 with this property cannot be a atomic formula . Let us suppose that the function v is in agreement with the function v_0 for any $a \in W$, for any $w \in W_0$ and for any formula containing less than α_0 logical symbols. Suppose that $w_0 \in W_0$ is a maximal element in the set $\{ w \in W_0 \mid v(\alpha_0, a_0, w) \neq v_0(\alpha_0, a_0, w)\}$. If $\alpha_0 = \beta \wedge \gamma$, then $v(\beta, a_0, w_0) = v_0(\beta, a_0, w_0)$ and $v(\gamma, a_0, w_0) = v_0(\gamma, a_0, w_0)$, thus $v(\alpha_0, a_0, w_0) = v(\beta \wedge \gamma, a_0, w_0) = \min\{v(\beta, a_0, w_0), v(\gamma, a_0, w_0)\} = \min\{v_0(\beta, a_0, w_0), v_0(\gamma, a_0, w_0)\} = v_0(\beta \wedge \gamma, a_0, w_0) = v_0(\alpha_0, a_0, w_0)$ - a contradiction. If $\alpha_0 = \beta \rightarrow \gamma$, $v(\alpha_0, a_0, w_0) = 1$ and $v_0(\alpha_0, a_0, w_0) = \phi$, then there exists $a_1 \geqslant a_0$ and $z \in W_0$ such that $z \geqslant_1 w_0$, $v_0(\beta, a_1, z) = 1$ and $v_0(\gamma, a_1, z) = \phi$. This is not possible for $z = w_0$ because then $v(\beta, a_1, z) = v(\beta, a_1, w_0) = 1$ and $v(\gamma, a_1, z) = v(\gamma, a_1, z) = \phi$, thus $v(\alpha_0, a_1, w_0) = \phi$ contrary to the assumption. Therefore there must be $z >_1 w_0$. Hence $v(\alpha_0, a_0, z) = v_0(\alpha_0, a_0, z)$, because w_0 is maximal in the set $\{ w \in W_0 \mid v(\alpha_0, a_0, w) \neq v_0(\alpha_0, a_0, w)\}$. Hence it follows that $v_0(\alpha_0, a_0, z) = 1$ - a contradiction. Let us now suppose that $\alpha_0 = \beta \rightarrow \gamma$, $v(\alpha_0, a_0, w_0) = \phi$ and $v_0(\alpha_0, a_0, w_0) = 1$. Then there exists $a_1 \in W$ and $z \in W_1$ such that $a_1 \geqslant a_0$ and $z \geqslant_1 w_0$, $v(\beta, a_1, z) = 1$, $v(\gamma, a_1, z) = \phi$. Let z_0 be a maximal element in the set $\{ z \in W_1 \mid z \geqslant_1 w_0, v(\beta, a_1, z) = 1, v(\gamma, a_1, z) = \phi\}$. If $z_0 \in W_0$, then $v(\beta, a_1, z_0) = 1$,

$v(\gamma, a_1, z_o) = \Phi$, hence $v_o(\alpha_o, a_1, w_o) = \Phi$ and so $v(\alpha_o, a_o, w_o) = \Phi$ - a contradiction. If $z_o \in W_1 - W_o$, then, on the strength of Lemma 3.1, $v(\alpha_o, a_1, z_o) = \min\{v(\alpha_o, a_o, u) \mid z_o \prec_1 u\}$ $= \Phi$. Then there exists u_o such that $z_o \prec_1 u_o$ and $v(\alpha_o, a_1, u_o) = \Phi$. Hence there must $a_2 \in W$ and $z \in W_1$ such that $a_2 \geqslant a_1$ and $z \quad _1 u_o$, $v(\quad , a_2, z) = 1$ and $v(\quad , a_2, z) = 0$. Thus $z >_1 z_o$, which contradicts the assumption that z_o is maximal. For $\alpha_o = \neg\beta$, the expected contradiction can be obtained by a similar reasoning. Let $\alpha_o = \forall x \beta$, $v(\alpha_o, a_o, w_o) = 1$ and $v_o(\alpha_o, a_o, w_o) = \Phi$. Then for any $a_1 \geqslant a_o$ and for any $\xi \in V(a_1)$, $v(\beta\bar{\xi}, a_1, w_o) = 1$, and there exists $a_1 \geqslant a_o$ and $\xi_o \in V(a_1)$ such that $v_o(\beta\bar{\xi}_o, a_1, w_o) = \Phi$. Thus on the strength of induction hipothesis $v_o(\beta\bar{\xi}_o, a_1, w_o) = v(\beta\bar{\xi}_o, a_1, w_o) = \Phi$, from this $v(\alpha_o, a_o, w_o) = \Phi$, which contradicts the assumption. It is clear that the expected contradiction can be obtained by a similar reasoning when $\alpha_o = \forall x \beta$, $v(\alpha_o, a_o, w_o) = \Phi$ and $v_o(\alpha_o, a_o, w_o) = 1$. Finally, let us assume that $\alpha_o = \exists x \beta$, $v(\alpha_o, a_o, w_o) = 1$ and $v_o(\alpha_o, a_o, w_o) = \Phi$. Then for any $\xi \in V(a_o)$, $v_o(\beta\bar{\xi}, a_o, w_o) = \Phi$ and there exists $\xi_o \in V(a_o)$ such that $v(\beta\bar{\xi}_o, a_o, w_o) = 1$. So by the induction hipothesis for $\xi_o \in V(a_o)$, $v(\beta\bar{\xi}_o, a_o, w_o) = v(\beta\bar{\xi}_o, a_o, w_o) = 1$, thus $v(\alpha_o, a_o, w_o) = v_o(\alpha_o, a_o, w_o) = 1$ - a contradiction. We omit the proof of the remaining part since it is completly similar. Q. E. D.

THEOREM 3.1. Let $\mathcal{W}_1 = \langle W_1, \leqslant_1 \rangle$ be a strongly atomic Kripke frame satisfying the increasing sequences condition. Let $\mathcal{W}_1^o = \langle W_1^o, \leqslant_1^o \rangle$ be a Kripke frame. Suppose that there exists a function h: $W_1^o \mapsto W_1$ such that:

(i) h is one-to-one,

(ii) for each $w_1, w_2 \in W_1^o$, $w_1 \leqslant_1^o w_2$ if and only if $h(w_1) \leqslant_1$

$h(w_2)$,

(iii) for each $w_1 \in W_1$, if w_1 is a maximal element of $_1$, then there exists u W_1^o such that $h(u) = w_1$.

Then $E(\mathcal{W}, v, \mathcal{W}_1) \cap FOR_{\{\wedge, \rightarrow, \neg, \forall, \exists\}} \subseteq E(\mathcal{W}, v, \mathcal{W}_1^o) \cap FOR_{\{\wedge, \rightarrow, \neg, \forall, \exists\}}$.

PROOF. It follows from Lemma 3.2.Q.E.D.

THEOREM 3.2. Suppose that the conditions of Theorem 3.1 hold. Let $f: \mathcal{W} \rightarrow \mathcal{W}^o$ be a embedding of $\mathcal{W} = W, \leqslant\rangle$ into $\mathcal{W}^o = \langle W^o, \leqslant^o\rangle$ and g be a function from $\bigcup\{V(a) \mid a \in W\}$ to $\bigcup\{V^o(b) \mid b \in W^o\}$ such that $g(V(a)) = V^o(f(a))$ for each $a \in W$. Then $E(\mathcal{W}, v, \mathcal{W}_1) \cap For_{\{\wedge, \rightarrow, \neg, \forall, \exists\}} \subseteq E(\mathcal{W}^o, v^o, \mathcal{W}_1^o) \cap For_{\{\wedge, \rightarrow, \neg, \forall, \exists\}}$.

PROOF. By Theorems 3.1 and 2.5.Q.E.D.

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SEMANTYKI TYPU KRIPKE'GO DLA POŚREDNICH SKWANTYFIKOWANYCH LOGIK

STRESZCZENIE

W pracy wyróżniamy uogólnione Kripke struktury oraz strukturalnie uogólnione Kripke struktury, następnie badamy pewne własności tych dwóch typów semantyk.