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Connections between connected topological spaces on the set of positive integers

Research Article

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Abstract: In this paper we introduce a connected topology $\mathcal{T}$ on the set $\mathbb{N}$ of positive integers whose base consists of all arithmetic progressions connected in Golomb's topology. It turns out that all arithmetic progressions which are connected in the topology $\mathcal{T}$ form a basis for Golomb's topology. Further we examine connectedness of arithmetic progressions in the division topology $\mathcal{T}'$ on $\mathbb{N}$ which was defined by Rizza in 1993. Immediate consequences of these studies are results concerning local connectedness of the topological spaces $(\mathbb{N}, \mathcal{T})$ and $(\mathbb{N}, \mathcal{T}')$.

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1. Preliminaries

The letters $\mathbb{Z}, \mathbb{N}$ and $\mathbb{N}_0$ denote the sets of integers, positive integers, and non-negative integers, respectively. For each set A we use $\operatorname{cl} A$ to denote its closure. The symbol $\Theta(a)$ denotes the set of all prime factors of $a \in \mathbb{N}$. For all $a, b \in \mathbb{N}$, we use (a, b) and $\operatorname{lcm}(a, b)$ to denote the greatest common divisor of a and b and the least common multiple of a and b, respectively. Moreover, for all $a, b \in \mathbb{N}$, $\{an+b\}$ and $\{an\}$ stand for the infinite arithmetic progressions

$$\{an+b\} \stackrel{\text{df}}{=} a \cdot \mathbb{N}_0 + b \qquad \text{and} \qquad \{an\} \stackrel{\text{df}}{=} a \cdot \mathbb{N}.$$

For basic results and notions concerning topology and number theory we refer the reader to the monographs of Engelking [3] and LeVeque [7], respectively.

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2. Introduction

In 1955 Furstenberg [4] defined the base of a topology $\mathcal{T}_F$ on $\mathbb{Z}$ by means of all arithmetic progressions and gave an elegant topological proof of the infinitude of primes. In 1959 Golomb [5] presented a similar proof of the infinitude of primes using a topology $\mathcal{D}$ on $\mathbb{N}$ with the base $\mathcal{B}_G = \{\{an+b\} : (a,b)=1\}$ defined in 1953 by Brown [2]. Ten years later Kirch [6] defined a topology $\mathcal{D}'$ on $\mathbb{N}$, weaker than Golomb's topology $\mathcal{D}$, with the base $\mathcal{B}_K = \{\{an+b\} : (a,b)=1, b<a, a \text{ is square-free}\}$. Both topologies $\mathcal{D}$ and $\mathcal{D}'$ are Hausdorff, the set $\mathbb{N}$ is connected in these topologies and locally connected in the topology $\mathcal{D}'$, but it is not locally connected in the topology $\mathcal{D}$, see [5, 6]. Recently the author showed that the arithmetic progression $\{an+b\}$ is connected in Golomb's topology $\mathcal{D}$ if and only if $\Theta(a) \subset \Theta(b)$ [9, Theorem 3.3]. Moreover it was proved that all arithmetic progressions are connected in Kirch's topology $\mathcal{D}'$ [9, Theorem 3.5].

In 1993 Rizza [8] introduced the division topology $\mathcal{T}'$ on the set $\mathbb{N}_0$ of non-negative integers as follows: for $X \subset \mathbb{N}_0$ he put

$$g(X) = \operatorname{cl} X = \bigcup_{x\in X} D(x), \qquad \text{where} \quad D(x) = \{y \in \mathbb{N}_0 : y \mid x\}.$$

The mapping g defines a topology $\mathcal{T}'$ on $\mathbb{N}_0$. Rizza showed that the division topology $\mathcal{T}'$ is a T_0-topology and it is not a T_1-topology. Moreover, the topological space $(\mathbb{N}_0, \mathcal{T}')$ is compact and connected [8, Propositions 2–4]. It is easy to see that $\mathcal{T}'$ is the right topology of the set $\mathbb{N}_0$ ordered by division, see e.g. [3, p. 81], and the family $\{a \cdot \mathbb{N}_0 : a \in \mathbb{N}_0\}$ is a basis for this topology.

In this paper we introduce a connected topology $\mathcal{T}$ on the set $\mathbb{N}$ of positive integers whose base consists of all arithmetic progressions connected in Golomb's topology $\mathcal{D}$. It turns out that the topology $\mathcal{T}$ is not locally connected, but all arithmetic progressions which are connected in the topology $\mathcal{T}$ form a basis for Golomb's topology. Further, we characterize connectedness of arithmetic progressions in the division topology $\mathcal{T}'$ restricted to the set $\mathbb{N}$. An immediate consequence of this characterization is local connectedness of the space $(\mathbb{N}, \mathcal{T}')$.

3. A new topology and its properties

Take as a basis $\mathcal{B}$ for a topology $\mathcal{T}$ on $\mathbb{N}$ all arithmetic progressions which are connected in Golomb's topology $\mathcal{D}$, i.e.

$$\mathcal{B} = \big\{\{an+b\} : \Theta(a) \subset \Theta(b)\big\}. \tag{1}$$

Indeed, for each $a \in \mathbb{N}$ there is an arithmetic progression $\{an+a\} = \{an\} \in \mathcal{B}$ such that $a \in \{an\}$. Now let us fix progressions $\{a_1n+b_1\}, \{a_2n+b_1\} \in \mathcal{B}$ and choose arbitrary $x \in \{a_1n+b_1\} \cap \{a_2n+b_2\}$. Let $c = \operatorname{lcm}(a_1, a_2)$. Since $\Theta(a_1) \subset \Theta(b_1)$, $\Theta(a_2) \subset \Theta(b_2)$ and $x \in \{a_1n+b_1\} \cap \{a_2n+b_2\}$, $\Theta(a_1) \cup \Theta(a_2) \subset \Theta(x)$, whence $\Theta(c) \subset \Theta(x)$. Thus there is an arithmetic progression $\{cn+x\} \in \mathcal{B}$ such that $x \in \{cn+x\}$. Moreover, we can easily see that

$$\{cn+x\} \subset \{a_1n+b_1\} \cap \{a_2n+b_2\}.$$

So, $\mathcal{B}$ forms a basis for the topology $\mathcal{T}$ on $\mathbb{N}$. Observe that every nonempty open set, being a union of basis arithmetic progressions, must be infinite. Now we will show some properties of the topological space $(\mathbb{N}, \mathcal{T})$.

Proposition 3.1.
Every nonempty $\mathcal{T}$-closed set in $\mathbb{N}$ contains the element 1.

Proof. Let F be a $\mathcal{T}$-closed nonempty set. Then $U = \mathbb{N} \setminus F$ is $\mathcal{T}$-open and $U \neq \mathbb{N}$. If $1 \in U$, then by (1) there were an arithmetic progression $\{an+b\} \in \mathcal{B}$ such that $\Theta(a) \subset \Theta(b)$ and $1 \in \{an+b\} \subset U$. Therefore $b = 1$ and $a = 1$, a contradiction. So, $1 \in F$. □

Proposition 3.2.
The topological space $(\mathbb{N}, \mathcal{T})$ *is connected and compact.*

Proof. By Proposition 3.1 we cannot find two nonempty closed sets having empty intersection. So, $(\mathbb{N}, \mathcal{T})$ is connected. Since every nonempty $\mathcal{T}$-closed set contains 1, the intersection of every centered system of $\mathcal{T}$-closed sets is nonempty, see [1, Definition 6, p. 11; Proposition 2, p. 57]. Therefore $(\mathbb{N}, \mathcal{T})$ is compact. □

Proposition 3.3.
$\mathcal{T}$ is a T_0-topology and it is not a T_1-topology.

Proof. First we will show that $\mathcal{T}$ is a T_0-topology. Fix $x, y \in \mathbb{N}$ with $x \neq y$. If $x = 1$, then there is an arithmetic progression $\{yn\} \in \mathcal{B}$ such that $1 \notin \{yn\}$. Clearly $\{yn\}$ is $\mathcal{T}$-open and $y \in \{yn\}$. So, let $x \neq 1$. There is $k \in \mathbb{N}$ such that $x^k > y$. Hence there is an arithmetic progression $\{x^k n + x\} \in \mathcal{B}$ such that $x \in \{x^k n + x\}$ and $y \notin \{x^k n + x\}$.

Now suppose that $\mathcal{T}$ is a T_1-topology. Let $x = 1$ and $y \neq x$. If U is $\mathcal{T}$-open with $1 \in U$ and $y \notin U$, then $1 \notin \mathbb{N} \setminus U$ and $\mathbb{N} \setminus U$ is $\mathcal{T}$-closed. By Proposition 3.1, $\mathbb{N} \setminus U = \emptyset$, whence $U = \mathbb{N}$. So, $y \in U$, a contradiction. □

Theorem 3.4.
The arithmetic progression $\{an+b\}$ is connected in the topological space $(\mathbb{N}, \mathcal{T})$ if and only if $(a, b) = 1$.

Proof. Let $\mathcal{B}$ be the base of the topology $\mathcal{T}$, see (1). Fix $a, b \in \mathbb{N}$.

"Only if" part. Assume that $(a, b) \neq 1$. Then there is a prime number p such that $p \mid a$ and $p \mid b$. We will show that in this case *the arithmetic progression $\{an+b\}$ is $\mathcal{T}$-disconnected*. Since $p \mid a$, we obtain

$$\{an+b\} \subset \{pn+b\}. \tag{2}$$

Moreover, $\Theta(p) = \{p\} \subset \Theta(b)$, whence $\{pn+b\} \in \mathcal{B}$. Choose $t \in \mathbb{N} \setminus \{1\}$ such that $p^{t-1} \mid a$ and $p^t \nmid a$. Then for $k \in \{0, \dots, p^{t-1}-1\}$ the progressions $\{p^t n + (pk+b)\}$ are pairwise disjoint and $\mathcal{T}$-open (as elements of the basis $\mathcal{B}$) and it is easy to check that

$$\{pn+b\} = \bigcup_{k=0}^{p^{t-1}-1} \{p^t n + (pk+b)\}. \tag{3}$$

From (2) and (3), we have

$$\{an+b\} = \{an+b\} \cap \bigcup_{k=0}^{p^{t-1}-1} \{p^t n + (pk+b)\} = X \cup Y,$$

where

$$X = \{an+b\} \cap \{p^t n + b\}, \qquad Y = \bigcup_{k=1}^{p^{t-1}-1} \big(\{an+b\} \cap \{p^t n + (pk+b)\}\big).$$

Consequently, the arithmetic progression $\{an+b\}$ splits into two disjoint sets X and Y which are $\mathcal{T}$-open in $\{an+b\}$.

Now we will show that both sets X and Y are nonempty. Obviously, the number $b \in \{an+b\} \cap \{p^t n + b\} = X$, whence X is nonempty. Further, by (2), $a + b \in \{an+b\} \subset \{pn+b\}$, whence

$$a + b \in \{pn+b\} \cap \{an+b\}. \tag{4}$$

Since $p^t \nmid a$, we have $a + b \notin \{p^t n + b\}$. Hence

$$a + b \notin \{p^t n + b\} \cap \{an+b\} = X. \tag{5}$$

From conditions (4) and (5) we obtain $a+b \in Y$, whence Y is nonempty. We thus have proved that if $(a,b) \neq 1$, then the arithmetic progression $\{an+b\}$ is $\mathcal{T}$-disconnected, as claimed.

"If" part. Now assume that the condition $(a,b) = 1$ is satisfied. We will prove that *the set* $\{an+b\}$ *is not* $\mathcal{T}$*-disconnected*. Assume the contrary: there are two disjoint nonempty sets O_1 and O_2 which are $\mathcal{T}$-open in $\{an+b\}$ and such that $\{an+b\} = O_1 \cup O_2$. Hence there exist two $\mathcal{T}$-open sets U_1, U_2 such that

$$O_1 = U_1 \cap \{an+b\} \qquad \text{and} \qquad O_2 = U_2 \cap \{an+b\}.$$

Suppose that $b \in O_1$. (The case $b \in O_2$ is analogous.) Then $b \in U_1$, whence there is an arithmetic progression $\{a_1 n + b\} \in \mathcal{B}$ such that

$$\Theta(a_1) \subset \Theta(b) \qquad \text{and} \qquad \{a_1 n + b\} \subset U_1.$$

So, since $(a,b) = 1$, we have $(a, a_1) = 1$. Now we consider two cases.

Case 1: $p \in O_2$ *for some prime number* $p \in \{an+b\}$ *such that* $p > a_1$. Obviously, $p \in U_2$. Hence and since the set U_2 is $\mathcal{T}$-open, there is $k \in \mathbb{N}$ such that $\{p^k n + p\} \subset U_2$. From conditions $p \in \{an+b\}$ and $(a,b) = 1$ we conclude that $(p,a) = 1$. Therefore $(p^k, a) = 1$ and since $p > a_1$, we have $(p, a_1) = 1$, whence $(p^k, a_1) = 1$. Consequently, $(p^k, aa_1) = 1$. So, by the Chinese Remainder Theorem,

$$\emptyset \neq \{an+b\} \cap \{a_1 n + b\} \cap \{p^k n + p\} \subset \{an+b\} \cap U_1 \cap U_2 = O_1 \cap O_2,$$

which contradicts the assumption that $O_1 \cap O_2 = \emptyset$.

Case 2: $p \in O_1$ *for each prime number* $p \in \{an+b\}$ *such that* $p > a_1$. Let $x \in O_2$. Then $x \in U_2$ and, since U_2 is $\mathcal{T}$-open, there is an arithmetic progression $\{a_2 n + x\} \in \mathcal{B}$ such that

$$\Theta(a_2) \subset \Theta(x) \qquad \text{and} \qquad \{a_2 n + x\} \subset U_2. \tag{6}$$

Since $x \in \{an+b\}$ and $(a,b) = 1$, $(a,x) = 1$ and, by condition (6), we have $(a, a_2) = 1$. Moreover, by Dirichlet's theorem (on primes in arithmetic progressions) there is a prime number $p \in \{an+b\}$ such that $p > \max\{a_1, a_2\}$. So, $p \in O_1 \subset U_1$. Since the set U_1 is $\mathcal{T}$-open, there is $k \in \mathbb{N}$ such that $\{p^k n + p\} \subset U_1$. Obviously $(p,a) = 1$, whence $(p^k, a) = 1$. Since $p > a_2$, we have $(p^k, a_2) = 1$. Consequently $(p^k, aa_2) = 1$, and by the Chinese Remainder Theorem,

$$\emptyset \neq \{an+b\} \cap \{p^k n + p\} \cap \{a_2 n + x\} \subset \{an+b\} \cap U_1 \cap U_2 = O_1 \cap O_2,$$

which contradicts $O_1 \cap O_2 = \emptyset$. So, the assumption that the progression $\{an+b\}$ may be $\mathcal{T}$-disconnected was false. □

Using Theorem 3.4 we can easily see that every base of the topology $\mathcal{T}$ contains some disconnected arithmetic progression. Therefore the following corollary holds.

Corollary 3.5.
The topological space $(\mathbb{N}, \mathcal{T})$ *is not locally connected.*

4. The division topology on the set $\mathbb{N}$

Let $(\mathbb{N}, \mathcal{T}')$ be a topological subspace of the space $(\mathbb{N}_0, \mathcal{T}')$, where $\mathcal{T}'$ is the division topology defined by Rizza. Clearly, $(\mathbb{N}, \mathcal{T}')$ is compact, connected (every nonempty $\mathcal{T}'$-closed set in $\mathbb{N}$ contains the element 1), T_0 (but not T_1) topological space with the base

$$\mathcal{B}' = \{\{an\}\}. \tag{7}$$

So, every nonempty open set, being a union of basis arithmetic progressions, must be infinite. Moreover, $\mathcal{T}'$ is the right topology of the set $\mathbb{N}$ ordered by division.

Now we will show that the space $(\mathbb{N}, \mathcal{T}')$ is locally connected. To this end we will prove the following theorem.

Theorem 4.1.
Every arithmetic progression $\{an+b\}$ is connected in the topological space $(\mathbb{N}, \mathcal{T}')$.

Proof. Let $\mathcal{B}'$ be the base of the topology $\mathcal{T}'$ on $\mathbb{N}$, see (7). Fix $a, b \in \mathbb{N}$. First assume that $(a, b) = 1$. Since $\mathcal{T}' \subset \mathcal{T}$, by Theorem 3.4, the arithmetic progression $\{an+b\}$ is $\mathcal{T}'$-connected in $\mathbb{N}$. So, we can assume that $(a, b) \neq 1$.

Suppose that the arithmetic progression $\{an+b\}$ is $\mathcal{T}'$-disconnected, i.e. there are two disjoint nonempty sets O_1 and O_2, $\mathcal{T}'$-open in $\{an+b\}$ and such that $\{an+b\} = O_1 \cup O_2$. Then there exist two $\mathcal{T}'$-open sets U_1, U_2 such that

$$O_1 = U_1 \cap \{an+b\} \qquad \text{and} \qquad O_2 = U_2 \cap \{an+b\}. \tag{8}$$

Assume that $b \in O_1$. (The case $b \in O_2$ is analogous.) Then $b \in U_1$ and, since U_1 is $\mathcal{T}'$-open, there is an arithmetic progression $\{bn\} \in \mathcal{B}'$ such that $\{bn\} \subset U_1$. Let $(a, b) = d > 1$. Then there are relatively prime numbers $x, y \in \mathbb{N}$ such that $a = dx$ and $b = dy$. We consider two cases.

Case 1: $y = 1$. In this case $a = bx$ and since $\{bxn+b\} \subset \{bn\} \subset U_1$, we have by (8),

$$O_1 \;=\; U_1 \cap \{an+b\} \;=\; U_1 \cap \{bxn+b\} \;=\; \{bxn+b\} \;=\; \{an+b\},$$

which proves that $O_2 = \emptyset$, a contradiction. So, in this case the assumption that $\{an+b\}$ with $(a, b) \neq 1$ may be $\mathcal{T}'$-disconnected was false.

Case 2: $y \in \mathbb{N} \setminus \{1\}$. Since $O_2 \neq \emptyset$, there is $c \in O_2 \subset U_2$. Since U_2 is $\mathcal{T}'$-open, there is an arithmetic progression $\{cn\} \in \mathcal{B}'$ such that $\{cn\} \subset U_2$. Moreover, since $c \in \{an+b\}$, there is $n_1 \in \mathbb{N}_0$ such that $c = an_1 + b$. Now consider two arithmetic progressions $\{(xn_1+y)n\}$ and $\{xn+1\}$. Observe that $\{(xn_1+y)n\} \subsetneq \mathbb{N}$. If there were a prime number p with $p \,|\, (xn_1+y)$ and $p \,|\, x$, we would have had $p \,|\, y$, which contradicts $(x, y) = 1$. Hence, $(xn_1 + y, x) = 1$. By the Chinese Remainder Theorem there is $\alpha \in \{(xn_1+y)n\} \cap \{xn+1\}$, whence there are $k_1 \in \mathbb{N}$ and $k_2 \in \mathbb{N}_0$ such that

$$\alpha = (xn_1+y)k_1 = xk_2 + 1. \tag{9}$$

Put $\beta = ayk_2 + b$. Clearly,

$$\beta \in \{an+b\}, \tag{10}$$

$$\beta = dxyk_2 + b = bxk_2 + b = b(xk_2+1) \in \{bn\} \subset U_1. \tag{11}$$

Moreover, by (11) and (9),

$$\beta = b(xk_2+1) = b(xn_1+y)k_1 = dyxn_1k_1 + dy^2k_1 = (an_1+b)k_1y = ck_1y \in \{cn\} \subset U_2 \tag{12}$$

So, by (10)–(12) and (8) we have

$$\beta \in \{an+b\} \cap U_1 \cap U_2 = O_1 \cap O_2,$$

which contradicts the assumption that $O_1 \cap O_2 = \emptyset$. So, the progression $\{an+b\}$ with $(a, b) \neq 1$ is $\mathcal{T}'$-connected in $\mathbb{N}$. □

An immediate consequence of Theorem 4.1 is the following corollary.

Corollary 4.2.
The topological space $(\mathbb{N}, \mathcal{T}')$ is locally connected.

5. Comparison of connected topologies on $\mathbb{N}$

One can observe some interesting connections between four topologies considered in this paper. First, when we take topologies $\mathcal{T}$ and $\mathcal{T}'$ we obtain the following relation.

Proposition 5.1.
The topology $\mathcal{T}$ is stronger than the division topology $\mathcal{T}'$.

Proof. Since $\mathcal{B}' \subset \mathcal{B}$, see (1) and (7), every $\mathcal{T}'$-open set is $\mathcal{T}$-open, too. Now consider the arithmetic progression $\{4n+2\}$, which obviously is an element of the base $\mathcal{B}$. Observe that $2 \in \{4n+2\}$ and $\{2n\}$ is the smallest set of the base $\mathcal{B}'$ containing 2. But $\{2n\} \not\subset \{4n+2\}$, which proves that $\mathcal{T}' \subsetneq \mathcal{T}$. □

Second, the base of Golomb's topology $\mathcal{D}$ consists of all arithmetic progressions that are connected in the topology $\mathcal{T}$, and conversely, all arithmetic progressions connected in $\mathcal{T}$ form a basis for $\mathcal{D}$. And finally, the connections between topologies $\mathcal{T}$ and $\mathcal{T}'$ on $\mathbb{N}$ are analogous to the connections between Golomb's topology $\mathcal{D}$ and Kirch's topology $\mathcal{D}'$, namely, stronger topologies are connected but not locally connected and weaker topologies are both connected and locally connected.

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