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MAXIMUMS OF EXTRA STRONG ŚWIĄTKOWSKI FUNCTIONS

Paulina Szczuka

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ABSTRACT. In this paper, we present some results concerning functions which are represented as the maximum of two extra strong Świątkowski functions.

1. Preliminaries

We use mostly standard terminology and notation. The letters $\mathbb{R}$ and $\mathbb{N}$ denote the real line and the set of positive integers, respectively. The symbols $\mathrm{I}(a, b)$ and $\mathrm{I}[a, b]$ denote the open and the closed interval with endpoints a and b, respectively. For each $A \subset \mathbb{R}$, we use the symbol $\operatorname{Int} A$ to denote its interior.

Let I be an interval and $f\colon I \to \mathbb{R}$. We say that f is a *Darboux function* ($f \in \mathcal{D}$), if it maps connected sets onto connected sets. We say that f is a *quasi-continuous function* [2] at a point $x \in I$ if for all open sets $U \ni x$ and $V \ni f(x)$ we have $\operatorname{Int}\big(U \cap f^{-1}(V)\big) \neq \emptyset$. The symbols $\mathcal{C}(f)$, $\mathcal{C}^+(f)$, $\mathcal{C}^-(f)$, and $\mathcal{Q}(f)$ will stand for the set of points of continuity, right-hand continuity, left-hand continuity, and quasi-continuity of f, respectively. If $\mathcal{Q}(f) = I$, then we say that f is *quasi-continuous* ($f \in \mathcal{Q}$). We say that f is a *strong Świątkowski function* [3] ($f \in \acute{\mathscr{S}}_s$), if whenever $\alpha, \beta \in I$, $\alpha < \beta$, and $y \in \mathrm{I}\big(f(\alpha), f(\beta)\big)$, there is an $x_0 \in (\alpha, \beta) \cap \mathcal{C}(f)$ such that $f(x_0) = y$. We say that f is an *extra strong Świątkowski function* ($f \in \acute{\mathscr{S}}_{es}$), if whenever $\alpha, \beta \in I$, $\alpha < \beta$, and $y \in \mathrm{I}\big[f(\alpha), f(\beta)\big]$, there is an $x_0 \in [\alpha, \beta] \cap \mathcal{C}(f)$ such that $f(x_0) = y$. (Clearly

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$\acute{\mathscr{S}}_{es} \subset \acute{\mathscr{S}}_s \subset \mathcal{D} \cap \mathcal{Q}$ and both inclusions are proper.) Finally let

$$\mathcal{S}(f) = \bigcup \{(a,b) :\ f{\restriction}(a,b) \in \acute{\mathscr{S}}_s\} \quad \text{and} \quad \mathcal{U}(f) = \bigcup \{(a,b) :\ f{\restriction}(a,b) \in \acute{\mathscr{S}}_{es}\}.$$

Now assume that $f\colon \mathbb{R} \to \mathbb{R}$. If $A \subset \mathbb{R}$ and x is a limit point of A, then let

$$\overline{\lim}(f, A, x) = \overline{\lim_{t \to x, t \in A}} f(x).$$

Similarly we define $\overline{\lim}(f, A, x^-)$ and $\overline{\lim}(f, A, x^+)$. Moreover we write $\overline{\lim}(f, x)$ instead of $\overline{\lim}(f, \mathbb{R}, x)$, etc. Let $x \in \mathbb{R}$. If $x_n \to x$ and $x_n < x_{n+1} < x$ for all $n \in \mathbb{N}$, then we will write $x_n \nearrow x$ ($x \nwarrow x_n$). Similarly, if $x_n \to x$ and $x_n > x_{n+1} > x$ for all $n \in \mathbb{N}$, then we will write $x_n \searrow x$ ($x \swarrow x_n$).

2. Introduction

In 1992 T. Natkaniec proved the following result [5, Proposition 3].

Theorem 2.1. *For every function f the following conditions are equivalent:*

a) *there are quasi-continuous functions g_1 and g_2 with $f = \max\{g_1, g_2\}$,*
b) *the set $\mathbb{R} \setminus \mathcal{Q}(f)$ is nowhere dense, and $f(x) \leq \overline{\lim}(f, \mathcal{C}(f), x)$ for each $x \in \mathbb{R}$.*

(In 1996 this theorem was generalized by J. Borsík for functions defined on regular second countable topological spaces [1].) He remarked also that if a function f can be written as the maximum of Darboux quasi-continuous functions, then

$$f(x) \leq \min\{\overline{\lim}(f, \mathcal{C}(f), x^-), \overline{\lim}(f, \mathcal{C}(f), x^+)\} \qquad \text{for each} \quad x \in \mathbb{R}, \tag{1}$$

and asked whether the following conjecture is true [5, Remark 3].

Conjecture 2.2. *If f is a function such that $\mathbb{R} \setminus \mathcal{Q}(f)$ is nowhere dense and condition (1) holds, then there are Darboux quasi-continuous functions g_1 and g_2 with $f = \max\{g_1, g_2\}$.*

In 1999 A. Maliszewski showed that this conjecture is false, and proved some facts about the maximums of Darboux quasi-continuous functions [4]. However, the problem of characterization of the maximums of Darboux quasi-continuous functions is still open.

In 2002 I proved the following theorem [6, Theorem 4.1].

Theorem 2.3. *For every function f the following conditions are equivalent:*

a) *there are functions $g_1, g_2 \in \acute{\mathscr{S}}_s$ with $f = \max\{g_1, g_2\}$,*
b) *the set $\mathcal{S}(f)$ is dense in $\mathbb{R}$, and*

$$f(x) \leq \min\{\overline{\lim}(f, \mathcal{C}(f), x^+), \overline{\lim}(f, \mathcal{C}(f), x^-)\} \qquad \textit{for each} \quad x \in \mathbb{R}.$$

In this paper we examine even smaller class of functions, namely the family $\acute{\mathscr{S}}_{es}$ of extra strong Świątkowski functions, and we present conditions which are necessary to represent the function f as the maximum of two extra strong Świątkowski functions (Theorem 4.1).

3. Auxiliary lemmas

The proofs of next three lemmas will be omitted. Lemma 3.1 we can prove by the same way as [6, Lemma 3.2], the proof of Lemma 3.2 is similar to the proof of [6, Lemma 3.3], and the proof of Lemma 3.3 is almost the same as the proof of [6, Theorem 3.5].

Lemma 3.1. *Let $a_0 < a_1 < a_2$. If $f\restriction[a_{i-1}, a_i] \in \acute{\mathscr{S}}_{es}$ for $i \in \{1, 2\}$ and $a_1 \in \mathcal{C}(f)$, then $f\restriction[a_0, a_2] \in \acute{\mathscr{S}}_{es}$.*

Lemma 3.2. *If I is a compact interval and $I \subset \mathcal{U}(f)$, then $f\restriction I \in \acute{\mathscr{S}}_{es}$.*

Lemma 3.3. *Let $g_1, g_2 \colon \mathbb{R} \to \mathbb{R}$ and $f = \max\{g_1, g_2\}$. If the sets $\mathcal{U}(g_1)$ and $\mathcal{U}(g_2)$ are dense in $\mathbb{R}$, then $\mathcal{U}(f)$ is dense in $\mathbb{R}$, too.*

Lemma 3.4. *Let $f \colon \mathbb{R} \to \mathbb{R}$, $c \in \mathbb{R}$, and $I = (a, b)$ be an open interval. If there are extra strong Świątkowski functions g_1 and g_2 with $f = \max\{g_1, g_2\}$ and $f(x) < c$ for each $x \in I \cap \mathcal{C}(f)$, then $f(x) < c$ for each $x \in I$.*

Proof. Let $f = \max\{g_1, g_2\}$, where $g_1, g_2 \in \acute{\mathscr{S}}_{es}$. Assume that

$$f(x) < c \qquad \text{for each} \quad x \in (a, b) \cap \mathcal{C}(f). \tag{2}$$

Since each extra strong Świątkowski function is quasi-continuous, according to Theorem 2.1, for each $x \in (a, b)$ we have $f(x) \leq \overline{\lim}\big(f, \mathcal{C}(f), x\big)$. So,

$$f(x) \leq c \qquad \text{for each} \quad x \in (a, b). \tag{3}$$

Now suppose that

$$f(x_0) = c \qquad \text{for some} \quad x_0 \in (a, b). \tag{4}$$

We will show that condition (4) is impossible. Assume that $f(x_0) = g_1(x_0)$. (The case $f(x_0) = g_2(x_0)$ is analogous.) Let $a < z < x_0$. Since $g_1 \in \acute{\mathscr{S}}_{es}$, there is an $x_1 \in [z, x_0] \cap \mathcal{C}(g_1)$ such that $g_1(x_1) = c$. Let $\varepsilon > 0$. Since $x_1 \in \mathcal{C}(g_1)$, there is a $\delta > 0$ such that

$$|g_1(x) - g_1(x_1)| < \varepsilon \qquad \text{for each} \quad x \in (x_1 - \delta, x_1 + \delta) \cap (a, b).$$

By (3) and since $f = \max\{g_1, g_2\}$, we have $g_1(x_1) = c \geq f(x_1) \geq g_1(x_1)$. Hence

$$g_1(x_1) = f(x_1) \geq f(x) \geq g_1(x) \qquad \text{for each} \quad x \in (a, b).$$

This yields

$$|f(x) - f(x_1)| \leq |g_1(x) - g_1(x_1)| < \varepsilon \qquad \text{for each} \quad x \in (x_1 - \delta, x_1 + \delta) \cap (a, b),$$

whence $x_1 \in \mathcal{C}(f)$. Therefore $c = f(x_1) < c$, a contradiction. □

4. Main result

Theorem 4.1. *Assume that $f\colon \mathbb{R} \to \mathbb{R}$ and there are extra strong Świątkowski functions g_1 and g_2 with $f = \max\{g_1, g_2\}$. Then the set $\mathcal{U}(f)$ is dense in $\mathbb{R}$ and for each $a \notin \mathcal{C}(f)$ at least one of the conditions below must be satisfied:*

a) *there are sequences $(x_n), (y_n) \subset \mathcal{C}(f)$ such that $x_n \nearrow a \swarrow y_n$, $f(x_n) \geq f(a)$, and $f(y_n) \geq f(a)$ for each $n \in \mathbb{N}$,*

or

b) *$a \in \mathcal{C}^+(f)$, $\underline{\lim}(f, a^-) = f(a)$, and for each $y > f(a)$ there is a $\delta > 0$ such that $\max\{f, y\}\restriction(a - \delta, a) \in \acute{\mathcal{S}}_{es}$,*

or

c) *$a \in \mathcal{C}^-(f)$, $\underline{\lim}(f, a^+) = f(a)$, and for each $y > f(a)$ there is a $\delta > 0$ such that $\max\{f, y\}\restriction(a, a + \delta) \in \acute{\mathcal{S}}_{es}$.*

P r o o f. Let $f = \max\{g_1, g_2\}$, where $g_1, g_2 \in \acute{\mathcal{S}}_{es}$. By Lemma 3.3, the set $\mathcal{U}(f)$ is dense in $\mathbb{R}$. Fix an $a \notin \mathcal{C}(f)$ and suppose that condition a) is not satisfied.

Assume that there is a $\tau > 0$ such that $f(x) < f(a)$ for all $x \in (a, a+\tau) \cap \mathcal{C}(f)$. We will prove that condition b) holds. Let $I = (a, a + \tau)$. By Lemma 3.4,

$$f(x) < f(a) \qquad \text{for each} \quad x \in I. \tag{5}$$

Assume that $f(a) = g_1(a)$. (The case $f(a) = g_2(a)$ is analogous.) First we claim that $a \in \mathcal{C}(g_1)$.

Suppose the contrary. Since $g_1 \in \acute{\mathcal{S}}_{es}$ and $a \notin \mathcal{C}(g_1)$, there is an $x_0 \in I \cap \mathcal{C}(g_1)$ such that $g_1(x_0) = g_1(a)$. But, by (5)

$$f(a) = g_1(a) = g_1(x_0) \leq f(x_0) < f(a),$$

a contradiction. So, $a \in \mathcal{C}(g_1)$ as claimed.

Now we will show that $a \in \mathcal{C}^+(f)$. By condition (5), and since $a \in \mathcal{C}^+(g_1)$ and $f = \max\{g_1, g_2\}$, we have

$$\underline{\lim}(f, a^+) \geq \underline{\lim}(g_1, a^+) = g_1(a) = f(a) \geq \overline{\lim}(f, a^+).$$

Further we will prove that $\underline{\lim}(f, a^-) = f(a)$. Conditions $f = \max\{g_1, g_2\}$ and $a \in \mathcal{C}(g_1)$ imply

$$\underline{\lim}(f, a^-) \geq \underline{\lim}(g_1, a^-) = g_1(a) = f(a).$$

Now suppose that $\underline{\lim}(f, a^-) > f(a)$. Define

$$\varepsilon = \frac{\underline{\lim}(f, a^-) - f(a)}{2} > 0.$$

Since $a \in \mathcal{C}(g_1)$, there is a $\delta > 0$ such that

$$f(x) \geq f(a) + \varepsilon \quad \text{and} \quad g_1(x) < f(a) + \varepsilon \qquad \text{for each} \quad x \in (a - \delta, a).$$

Hence $f = g_2$ on $(a-\delta, a)$, and

$$\underline{\lim}(g_2, a^-) = \underline{\lim}(f, a^-) > f(a) \geq g_2(a).$$

So, $g_2 \notin \mathcal{D} \supset \acute{\mathcal{S}}_{es}$, a contradiction. Consequently, $\underline{\lim}(f, a^-) = f(a)$.

It remains to prove that for each $y > f(a)$ there is a number $\delta > 0$ such that $\max\{f, y\}\restriction(a-\delta, a) \in \acute{\mathcal{S}}_{es}$. Fix a $y > f(a)$. Since $a \in \mathcal{C}(g_1)$, there is a $\delta > 0$ such that

$$|g_1(x) - g_1(a)| < y - f(a) \qquad \text{for each} \quad x \in (a-\delta, a).$$

Since $f = \max\{g_1, g_2\}$, for each $x \in (a-\delta, a)$ the inequality $f(x) \geq y$ implies $f(x) = g_2(x)$. Therefore $\max\{f, y\} = \max\{g_2, y\}$ on $(a-\delta, a)$. Taking into account that $g_2 \in \acute{\mathcal{S}}_{es}$ and the maximal class with respect to maximums for the family of extra strong Świątkowski functions (i.e., the family of all functions whose maximum with every element of $\acute{\mathcal{S}}_{es}$ belongs to $\acute{\mathcal{S}}_{es}$) consists of constant functions only [7, Corollary 3.6], we conclude that $\max\{g_2, y\} \in \acute{\mathcal{S}}_{es}$. Consequently,

$$\max\{f, y\}\restriction(a-\delta, a) = \max\{g_2, y\}\restriction(a-\delta, a) \in \acute{\mathcal{S}}_{es},$$

which completes the proof of condition b).

Analogously we can show that if there is a $\tau > 0$ such that $f(x) < f(a)$ for each $x \in (a-\tau, a) \cap \mathcal{C}(f)$, then condition c) holds. This completes the proof. □

Now we will show that the condition: "for each $a \notin \mathcal{C}(f)$ there are sequences $(x_n), (y_n) \subset \mathcal{C}(f)$ such that $x_n \nearrow a \swarrow y_n$, $f(x_n) \geq f(a)$, and $f(y_n) \geq f(a)$ for each $n \in \mathbb{N}$" is not necessary for a function f to be the maximum of two extra strong Świątkowski functions.

Example 4.2. There is a function $f\colon \mathbb{R} \to \mathbb{R}$ which not satisfied condition a) of Theorem 4.1 and which is the maximum of two extra strong Świątkowski functions.

C o n s t r u c t i o n. Define

$$f(x) = \begin{cases} \max\{\sin x^{-1}, x\} & \text{if } x < 0, \\ -x & \text{if } x \geq 0. \end{cases}$$

Observe that 0 is the only point of discontinuity of f. Since $f(x) < f(0)$ for each $x \in (0, +\infty)$, condition a) of Theorem 4.1 is not satisfied. Now define $g_1(x) = -|x|$ and

$$g_2(x) = \begin{cases} \sin x^{-1} & \text{if } x < 0, \\ -\frac{1}{2} & \text{if } x = 0, \\ \min\{\sin x^{-1}, -x\} & \text{if } x > 0. \end{cases}$$

Clearly $f = \max\{g_1, g_2\}$, and $g_1, g_2 \in \acute{\mathcal{S}}_{es}$. □

The next example shows a difference between maximums of strong Świątkowski functions and maximums of extra strong Świątkowski functions.

Example 4.3. There is a strong Świątkowski function $f\colon \mathbb{R} \to \mathbb{R}$ which cannot be written as the maximum of two extra strong Świątkowski functions.

C o n s t r u c t i o n. Define

$$f(x) = \begin{cases} \sin x^{-1} - |x| & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Observe that $0 \notin \mathcal{C}(f)$ and conditions a), b), and c) of Theorem 4.1 are not satisfied, whence f cannot be written as the maximum of two extra strong Świątkowski functions. But $f \in \acute{\mathscr{S}}_s$ and $f = \max\{f, f\}$, therefore f is the maximum of two strong Świątkowski functions. □

Finally we would like to present the following conjecture.

CONJECTURE 4.4. *If the set* $\mathcal{U}(f)$ *is dense in* $\mathbb{R}$ *and for each* $a \notin \mathcal{C}(f)$ *a function* $f\colon \mathbb{R} \to \mathbb{R}$ *satisfies condition* a) *or* b) *or* c) *of* Theorem 4.1*, then there are extra strong Świątkowski functions* g_1 *and* g_2 *with* $f = \max\{g_1, g_2\}$.

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Department of Mathematics
Kazimierz Wielki University
pl. Weyssenhoffa 11
PL–85–072 Bydgoszcz
POLAND
E-mail: paulinaszczuka@wp.pl