ZESZYTY NAUKOWE WYŻSZEJ SZKOŁY PEDAGOGICZNEJ w BYDGOSZCZY
Problemy Matematyczne 1985 z. 7

MATERIAŁY Z MIEDZYNARODOWEJ KONFERENCJI TEORII FUNKCJI RZECZYWISTYCH

MAREK BALCERZAK
Uniwersytet Łódzki

A GENERALIZATION OF THE THEOREM OF MAULDIN

Let X be a metric space and let $\mathcal{J}$ be a proper σ-ideal of subsets of X. It will be assumed that all singletons $\{x\}$, $x \in X$, belong to $\mathcal{J}$.

Denote by $\Phi_0(\mathcal{J})$ the family of all real-valued functions defined on X whose set of points of discontinuity belongs to $\mathcal{J}$. For each ordinal α, $0 < \alpha \leq \omega_1$, $\Phi_\alpha(\mathcal{J})$ be the family of all pointwise limits of sequences which terms are taken form $\bigcup_{\beta<\alpha} \Phi_\beta(\mathcal{J})$. The first number α such that $\Phi_\alpha(\mathcal{J}) = \Phi_{\omega_1}(\mathcal{J})$ will be called the Baire order of the σ-ideal $\mathcal{J}$.
The generalized Baire classes $\Phi_\alpha(\mathcal{J})$ were considered by Mauldin (see [6],[7],[8]).

In [2] Kuratowski proved that if X complete and separable, and $\mathcal{J}$ denotes the σ-ideal of all sets of the first category, then the order of $\mathcal{J}$ is 1. In [7] Mauldin proved that if $\mathcal{J}$ denotes the σ-ideal of all subsets of $[0,1]$ of the Lebesgue measure zero, then the order of $\mathcal{J}$ is ω_1.
We have obtained the following generalization of this result:

Theorem 1. Let X be a perfect metric space, complete and separable. Let $\mathcal{J}_0$ be a σ-ideal of subsets of X such that

(1) there is a compact set $X_0 \subseteq X$ such that $X_0 \notin \mathcal{J}_0$,

(2) for each countable set $A \subseteq X$ there is a G_δ set B such that $A \subseteq B \in \mathcal{J}_0$.

Then for each σ-ideal $\mathcal{J}$ such that $\mathcal{J} \subseteq \mathcal{J}_0$ the order of $\mathcal{J}$ is ω_1.

Remarks and problems. (a) In the case when $X = [0,1]$ and $\mathcal{J} = \mathcal{J}_0$ is the ideal of sets of the measure zero, we obtain

Mauldin's result.

(b) The condition (1) is fulfilled when X is locally compact. Indeed, then we put as X_o a compact set which is a closure of an open nonempty set . Can the condition (1) be omitted in the general case?

(c) In [9] Mycielski constructed a σ-ideal of subsets of the Cantor set C which satisfies the condition (2) . Since $X = C$ is compact, the condition (1) also holds. So Theorem 1 can be applied.

(d) Let X be such as in Theorem 1 and moreover let X be locally compact. Suppose that $\mathcal{J}$ is a σ-ideal of X with the order ω_1 .
Does there σ-ideal $\mathcal{J}_o$ exist such that $\mathcal{J} \subseteq \mathcal{J}_o$ and $\mathcal{J}_o$ fulfils the condition (2) ?

The proof of Theorem 1 is based on the method presented by Mauldin. A new element of the proof is the application of the topology $\tau(\mathcal{J})$ associated with the ideal $\mathcal{J}$. This topology was investigated by many authors (comp. [1], [4], [5], [9]). New properties of $\tau(\mathcal{J})$ which were used in the proof of Theorem 1 will be presented here.

Definition 1. For $A \subseteq X$ let $A^{(\mathcal{J})}$ be the set of all $x \in X$ such that $V \cap A \notin \mathcal{J}$ for every neighbourhood V of x.

In turns out that $A \to A^{(\mathcal{J})}$ satisfies all conditions of the operator of the derived set and it yields a topology $\tau(\mathcal{J})$.

Definition 2. A closed set $A, \emptyset \neq A \subseteq X$ will be called $\mathcal{J}$-perfect if and only if for each set V such that $V \cap A \neq \emptyset$, we have $V \cap A \notin \mathcal{J}$.

Proposition 1. A set A, $\emptyset \neq A \subseteq X$ is $\mathcal{J}$-perfect if and only if $A = A^{(\mathcal{J})}$.

(It means that $\mathcal{J}$-perfect sets coincide with perfect sets in the topology $\tau(\mathcal{J})$) .

Proposition 2. For each closed set $A \subseteq X$ there is a unique decomposition $A = B \cup C$ into dijoint sets B, C such that $B \in \mathcal{J}$, and $C = \emptyset$ or C is $\mathcal{J}$-perfect.

That is a generalization of the Cantor-Bendixson Theorem.

If denotes the σ-ideal of all countable sets, then we have the classic formulation. A similar result was obtained by Louveau in [3] .

REFERENCES

[1] Hashimoto H., On the*topology and its application, Fund. Math. 91 (1976) 5-10

[2] Kuratowski K., Sur les fonctions représentable analytiquement et les ensembles de premiere categorie, Fund. Math. 5 (1924), 75-86

[3] Louveau A., σ -ideaux engendrés par des ensembles fermes et théorems d'approximation, Trans. Amer. Math. Soc. 257 (1980), 143-169

[4] Marczewski E., Traczyk T., On developable sets and almost-limits points, Colloq. Math 8 (1961), 55-66

[5] Martin N.F.G., Generalized condensation points, Duke Math. J. 28 (1961), 507-514

[6] Mauldin R.D., σ -ideals and related Baire systems, Fund. Math. 71 (1971), 171-177

[7] Mauldin R.D., The Baire order of the functions continuous almost everywhere, Proc. Amer. Math. Soc. 41 (1973), 535--540

[8] Mauldin R.D., The Baire order of the function continuous almost everywhere II, Proc. Amer. Math. Soc. 51 (1975), 371-377

[9] Mycielski J., Some new ideals of sets on the real line, Colloq. Math., 20 (1969), 71-76