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The closures of arithmetic progressions in the common division topology on the set of positive integers

Research Article

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Abstract: In this paper we characterize the closures of arithmetic progressions in the topology $\mathcal{T}$ on the set of positive integers with the base consisting of arithmetic progressions $\{an+b\}$ such that if the prime number p is a factor of a, then it is also a factor of b. The topology $\mathcal{T}$ is called the common division topology.

MSC: 54A05, 11B25, 11A41, 11A05

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1. Preliminaries

The letters $\mathbb{N}, \mathbb{N}_0$ and $\mathcal{P}$ denote the sets of positive integers, non-negative integers and primes, respectively. For a set A we use the symbol $\operatorname{cl} A$ to denote the closure of A. The symbol $\Theta(a)$ denotes the set of all prime factors of $a \in \mathbb{N}$. For all $a, b \in \mathbb{N}$, we use (a, b) and $\operatorname{lcm}(a, b)$ to denote the greatest common divisor of a and b and the least common multiple of a and b, respectively. Moreover, for all $a, b \in \mathbb{N}$, the symbols $\{an+b\}$ and $\{an\}$ stand for the infinite arithmetic progressions:

$$\{an+b\} \stackrel{\text{df}}{=} a \cdot \mathbb{N}_0 + b \qquad \text{and} \qquad \{an\} \stackrel{\text{df}}{=} a \cdot \mathbb{N}.$$

Hence, clearly, $\{an\} = \{an+a\}$. For the basic results and notions concerning topology and number theory we refer the reader to the monographs of Kelley [4] and LeVeque [6], respectively.

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2. Introduction

In 1955 Furstenberg [2] defined the base of a topology on the set of integers by means of all arithmetic progressions and gave an elegant topological proof of the infinitude of primes. In 1959 Golomb [3] presented a similar proof of the infinitude of primes using a topology $\mathcal{D}$ on $\mathbb{N}$ with the base $\mathcal{B}_G = \big\{\{an+b\} : (a,b) = 1\big\}$ defined in 1953 by Brown [1]. Ten years later Kirch [5] defined a topology $\mathcal{D}'$ on $\mathbb{N}$, weaker than Golomb's topology $\mathcal{D}$, with the base $\mathcal{B}_K = \{\{an+b\} : (a,b) = 1,\ a \text{ is square-free}\}$. Both topologies $\mathcal{D}$ and $\mathcal{D}'$ are Hausdorff, the set $\mathbb{N}$ is connected in these topologies and locally connected in the topology $\mathcal{D}'$, but it is not locally connected in the topology $\mathcal{D}$ (see [3, 5]). Moreover, the set $\mathbb{N}$ is semiregular in the stronger topology $\mathcal{D}$ and it is not semiregular in the weaker topology $\mathcal{D}'$ (see [10]).

In 1993 Rizza [7] introduced the division topology $\mathcal{T}'$ on $\mathbb{N}$ as follows: for $X \subset \mathbb{N}$ he put

$$g(X) = \operatorname{cl} X = \bigcup_{x\in X} D(x), \qquad \text{where} \quad D(x) = \{y \in \mathbb{N} : y\,|\,x\}.$$

The mapping g forms a topology $\mathcal{T}'$ on $\mathbb{N}$. It is easy to see that the family $\mathcal{B}' = \{\{an\}\}$ is a basis for this topology. In [9] the author defined the common division topology $\mathcal{T}$ on $\mathbb{N}$, stronger than the division topology $\mathcal{T}'$, with the base $\mathcal{B} = \{\{an+b\} : \Theta(a) \subset \Theta(b)\}$. Both topologies $\mathcal{T}$ and $\mathcal{T}'$ are T_0 and they are not T_1, the set $\mathbb{N}$ is connected in these topologies and locally connected in the topology $\mathcal{T}'$, but it is not locally connected in the topology $\mathcal{T}$ (see [7, 9]). Moreover, the set $\mathbb{N}$ is semiregular in the stronger topology $\mathcal{T}$ and it is not semiregular in the weaker topology $\mathcal{T}'$ (see [10]).

Since 2010 the author has examined properties of arithmetic progressions in the above four topologies. It was already shown that the base of Golomb's topology $\mathcal{D}$ consists of all arithmetic progressions that are connected in the common division topology $\mathcal{T}$, and conversely, all arithmetic progressions connected in $\mathcal{T}$ form a basis for $\mathcal{D}$ (see [9]). Moreover, it turned out that all arithmetic progressions are connected in topologies $\mathcal{D}'$ and $\mathcal{T}'$ (see [8, Theorem 3.5] and [9, Theorem 4.1], respectively). Recently the author gave a characterization of regular open arithmetic progressions in these topologies (see [10]).

In this paper we continue studies concerning properties of arithmetic progressions, namely, we characterize closures of arithmetic progressions in the common division topology $\mathcal{T}$ on $\mathbb{N}$. From now on we will only deal with the common division topology and to simplify the notation the symbol $\mathcal{T}$ will be omitted.

3. Main results

We start with two simple technical lemmas.

Lemma 3.1.

Assume that U is an open set. If $c \in U$, then there is an arithmetic progression $\{an+c\} \in \mathcal{B}$ such that $\{an+c\} \subset U$.

Proof. Let $c \in U$. Since the set U is open, there is an arithmetic progression $\{an+b\} \in \mathcal{B}$ such that $c \in \{an+b\} \subset U$ and $\Theta(a) \subset \Theta(b)$. So, $\{an+c\} \subset \{an+b\} \subset U$ and $\Theta(a) \subset \Theta(c)$. This implies that $\{an+c\} \in \mathcal{B}$. □

Lemma 3.2.

If $b_1 \equiv b \pmod a$, then $\operatorname{cl}\{an+b\} = \operatorname{cl}\{an+b_1\}$.

Proof. Without loss of generality we can assume that $b_1 < b$. Since $\{an+b\} \subset \{an+b_1\}$, we have $\operatorname{cl}\{an+b\} \subset \operatorname{cl}\{an+b_1\}$. So, it is sufficient to show the opposite implication.

Let $x \in \operatorname{cl}\{an+b_1\}$. Fix an open set U with $x \in U$. By Lemma 3.1, there is a basic arithmetic progression $\{cn+x\} \subset U$. Since $\{cn+x\}$ contains x and it is open, $\{cn+x\} \cap \{an+b_1\} \neq \emptyset$. Taking into account that a nonempty intersection of two infinite arithmetic progressions is an infinite arithmetic progression, we can conclude that the set $\{cn+x\}\cap\{an+b_1\}$ is infinite. Simultaneously, the set $\{an+b_1\}\setminus\{an+b\}$ is finite, which implies

$$\emptyset \neq \{an+b\}\cap\{cn+x\} \subset \{an+b\}\cap U.$$

This proves that $x \in \mathrm{cl}\{an+b\}$. □

The proof of next remark is evident.

Remark 3.3.

$\mathrm{cl}\{n+b\} = \mathbb{N}$ for each $b \in \mathbb{N}$.

From now on in all theorems of this paper we assume $a > 1$.

Theorem 3.4.

Assume $p \in \mathcal{P}$, $k \in \mathbb{N}$ and $b_1 \leq p^k$. If $b_1 \equiv b \,(\mathrm{mod}\, p^k)$, then $\mathrm{cl}\{p^k n+b\} = \{p^k n+b_1\}\cup(\mathbb{N}\setminus\{pn\})$. In particular,

(i) *the arithmetic progression $\{2n+1\}$ is closed,*

(ii) $\mathrm{cl}\{pn\} = \mathbb{N}$ *for each $p \in \mathcal{P}$, and*

(iii) *if the arithmetic progression $\{pn+b\}$ is open, then $\mathrm{cl}\{pn+b\} = \mathbb{N}$ for each $p \in \mathcal{P}$.*

Proof. First we will show that $\mathrm{cl}\{p^k n+b\} \subset \{p^k n+b_1\}\cup(\mathbb{N}\setminus\{pn\})$. Using the assumptions $b_1 \leq p^k$ and $b_1 \equiv b \,(\mathrm{mod}\, p^k)$, we obtain

$$\{p^k n+b\} \subset \{p^k n+b_1\} \subset \{p^k n+b_1\}\cup(\mathbb{N}\setminus\{pn\}).$$

If $(p,b)=1$, then $(p,b_1)=1$, too. Hence $\{p^k n+b_1\}\subset\mathbb{N}\setminus\{pn\}$ and the set $\mathbb{N}\setminus\{pn\} = \{p^k n+b_1\}\cup(\mathbb{N}\setminus\{pn\})$ is closed. This proves that $\mathrm{cl}\{p^k n+b\}\subset\{p^k n+b_1\}\cup(\mathbb{N}\setminus\{pn\})$. So, we can assume $p\,|\,b$. Then, obviously, $p\,|\,b_1$, whence $\{p^k n+b_1\}\subset\{pn\}$. Since

$$\{pn\}\setminus\{p^k n+b_1\} = \bigcup_{i=1}^{p^{k-1}}\{p^k n+ip\}\setminus\{p^k n+b_1\} = \bigcup_{i\in\{1,\dots,p^{k-1}\}\setminus\{b_1\}}\{p^k n+ip\}$$

and all arithmetic progressions $\{p^k n+ip\}$ are open, the set $\{p^k n+b_1\}\cup(\mathbb{N}\setminus\{pn\}) = \mathbb{N}\setminus(\{pn\}\setminus\{p^k n+b_1\})$ is closed. Consequently, $\mathrm{cl}\{p^k n+b\}\subset\{p^k n+b_1\}\cup(\mathbb{N}\setminus\{pn\})$.

Now we will show the opposite inclusion. Let $x\in\{p^k n+b_1\}\cup(\mathbb{N}\setminus\{pn\})$. We consider two cases.

Case 1: $x\in\{p^k n+b_1\}$. Since $b_1\equiv b\,(\mathrm{mod}\,p^k)$, by Lemma 3.2, $\mathrm{cl}\{p^k n+b\} = \mathrm{cl}\{p^k n+b_1\}$. So, $x\in\{p^k n+b_1\}\subset \mathrm{cl}\{p^k n+b_1\} = \mathrm{cl}\{p^k n+b\}$.

Case 2: $x\in\mathbb{N}\setminus\{pn\} = \bigcup_{d\in\{1,\dots,p-1\}}\{pn+d\}$. Then $x\in\{pn+d\}$ for some $d\in\{1,\dots,p-1\}$. Fix an open set U such that $x\in U$. By Lemma 3.1, there is an arithmetic progression $\{cn+x\}\in\mathcal{B}$ with $\{cn+x\}\subset U$ and $\Theta(c)\subset\Theta(x)$. Since $x\in\mathbb{N}\setminus\{pn\}$, we have $(p,x)=1$. So, $(p,c)=1$, too. Using the Chinese Remainder Theorem (CRT), we obtain

$$\emptyset\neq\{cn+x\}\cap\{p^k n+b\}\subset U\cap\{p^k n+b\},$$

whence $x\in\mathrm{cl}\{p^k n+b\}$.

Finally, observe that conditions (i) and (ii) are evident. Moreover, since the arithmetic progression $\{pn+b\}$ is open, we have $p\equiv b\,(\mathrm{mod}\,p)$. So, $\mathrm{cl}\{pn+b\} = \{pn\}\cup(\mathbb{N}\setminus\{pn\}) = \mathbb{N}$, whence condition (iii) holds, too. □

Theorem 3.5.

Let $a = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ be the prime power factorization of a. Then $\operatorname{cl}\{an+b\} = \bigcap_{i=1}^{k} \operatorname{cl}\{p_i^{\alpha_i} n + b\}$.

Proof. First observe

$$\{an+b\} = \bigcap_{i=1}^{k} \{p_i^{\alpha_i} n + b\}.$$

Hence $\operatorname{cl}\{an+b\} \subset \bigcap_{i=1}^{k} \operatorname{cl}\{p_i^{\alpha_i} n + b\}$.

Now we will show the opposite inclusion. Assume $x \in \bigcap_{i=1}^{k} \operatorname{cl}\{p_i^{\alpha_i} n + b\}$. Then, by Theorem 3.4, $x \in \bigcap_{i=1}^{k} \big(\{p_i^{\alpha_i} n + b_i\} \cup (\mathbb{N} \setminus \{p_i n\})\big)$, where $b_i \equiv b \,(\operatorname{mod} p_i^{\alpha_i})$ and $b_i \le p_i^{\alpha_i}$ for each $i \in \{1, \dots, k\}$. Fix an open set U such that $x \in U$. By Lemma 3.1, there is an arithmetic progression $\{cn+x\} \in \mathcal{B}$ with $\{cn+x\} \subset U$. Hence $\Theta(c) \subset \Theta(x)$. We consider three cases.

Case 1: $x \in \bigcap_{i=1}^{k} \{p_i^{\alpha_i} n + b_i\}$. By CRT, there is exactly one $s \in \mathbb{N}$ such that $1 \le s \le p_1^{\alpha_1} \dots p_k^{\alpha_k}$ and

$$\bigcap_{i=1}^{k} \{p_i^{\alpha_i} n + b_i\} = \{(p_1^{\alpha_1} \dots p_k^{\alpha_k}) n + s\} = \{an+s\}.$$

Since $b_i \equiv b \,(\operatorname{mod} p_i^{\alpha_i})$ and $b_i \le p_i^{\alpha_i}$ for each $i \in \{1, \dots, k\}$, we have $\{p_i^{\alpha_i} n + b\} \subset \{p_i^{\alpha_i} n + b_i\}$ for each $i \in \{1, \dots, k\}$. Hence

$$\{an+b\} = \bigcap_{i=1}^{k} \{p_i^{\alpha_i} n + b\} \subset \bigcap_{i=1}^{k} \{p_i^{\alpha_i} n + b_i\} = \{an+s\}.$$

So, $s \equiv b \,(\operatorname{mod} a)$. By Lemma 3.2, we obtain that $\operatorname{cl}\{an+b\} = \operatorname{cl}\{an+s\}$. Consequently, $x \in \{an+s\} \subset \operatorname{cl}\{an+s\} = \operatorname{cl}\{an+b\}$.

Case 2: $x \in \bigcap_{i=1}^{k} (\mathbb{N} \setminus \{p_i n\})$. Since $\bigcap_{i=1}^{k} (\mathbb{N} \setminus \{p_i n\}) = \mathbb{N} \setminus \bigcup_{i=1}^{k} \{p_i n\}$, we have $x \notin \bigcup_{i=1}^{k} \{p_i n\}$. So, $(p_i, x) = 1$ for each $i \in \{1, \dots, k\}$. Hence $(p_i, c) = 1$ for each $i \in \{1, \dots, k\}$, which implies $(a, c) = 1$. By CRT,

$$\emptyset \ne \{cn+x\} \cap \{an+b\} \subset U \cap \{an+b\}.$$

Consequently, $x \in \operatorname{cl}\{an+b\}$.

Case 3: There are a number $r \in \{1, \dots, k-1\}$ and a permutation $\{\sigma_1, \dots, \sigma_k\}$ of the set $\{1, \dots, k\}$ such that $x \in \bigcap_{i=1}^{r} \{p_{\sigma_i}^{\alpha_{\sigma_i}} n + b_{\sigma_i}\} \cap \bigcap_{i=r+1}^{k} (\mathbb{N} \setminus \{p_{\sigma_i} n\})$. By CRT, there is exactly one $s \in \mathbb{N}$ such that $1 \le s \le p_{\sigma_1}^{\alpha_{\sigma_1}} \dots p_{\sigma_r}^{\alpha_{\sigma_r}}$ and

$$\bigcap_{i=1}^{r} \{p_{\sigma_i}^{\alpha_{\sigma_i}} n + b_{\sigma_i}\} = \{(p_{\sigma_1}^{\alpha_{\sigma_1}} \dots p_{\sigma_r}^{\alpha_{\sigma_r}}) n + s\}.$$

So,

$$x \in \{(p_{\sigma_1}^{\alpha_{\sigma_1}} \dots p_{\sigma_r}^{\alpha_{\sigma_r}}) n + s\} \cap \left(\mathbb{N} \setminus \bigcup_{i=r+1}^{k} \{p_{\sigma_i} n\}\right).$$

Define $a_1 = p_{\sigma_1}^{\alpha_{\sigma_1}} \dots p_{\sigma_r}^{\alpha_{\sigma_r}}$ and $a_2 = p_{\sigma_{r+1}}^{\alpha_{\sigma_{r+1}}} \dots p_{\sigma_k}^{\alpha_{\sigma_k}}$. Then $(a_1, a_2) = 1$. Moreover, $(p_{\sigma_i}, x) = 1$ for each $i \in \{r+1, \dots, k\}$. Hence $(p_{\sigma_i}, c) = 1$ for each $i \in \{r+1, \dots, k\}$, which implies $(a_2, c) = 1$. Since $b_{\sigma_i} \equiv b \left(\operatorname{mod} p_{\sigma_i}^{\alpha_{\sigma_i}}\right)$ and $b_{\sigma_i} \le p_{\sigma_i}^{\alpha_{\sigma_i}}$ for each $i \in \{1, \dots, r\}$, we obtain that $\{p_{\sigma_i}^{\alpha_{\sigma_i}} n + b\} \subset \{p_{\sigma_i}^{\alpha_{\sigma_i}} + b_{\sigma_i}\}$ for each $i \in \{1, \dots, r\}$. So,

$$\{a_1 n + b\} = \bigcap_{i=1}^{r} \{p_{\sigma_i}^{\alpha_{\sigma_i}} n + b\} \subset \bigcap_{i=1}^{r} \{p_{\sigma_i}^{\alpha_{\sigma_i}} n + b_{\sigma_i}\} = \{a_1 n + s\}.$$

Hence

$$\{a_1n+s\}\cap\{a_2n+b\}=\{an+b\}. \tag{1}$$

Since $x\in\{a_1n+s\}$, we have $\{a_1n+s\}\cap\{cn+x\}\neq\emptyset$. It is known that nonempty intersection of two infinite arithmetic progressions is an infinite arithmetic progression. Therefore

$$\{a_1n+s\}\cap\{cn+x\}=\{dn+e\},\qquad \text{where}\quad d=\operatorname{lcm}(c,a_1).$$

Moreover, if $(c,a_2)=1$ and $(a_1,a_2)=1$, then $(d,a_2)=1$. So, by CRT and condition (1), we obtain

$$\emptyset\neq\{dn+e\}\cap\{a_2n+b\}\;=\;\{a_1n+s\}\cap\{cn+x\}\cap\{a_2n+b\}\;=\;\{cn+x\}\cap\{an+b\}\;\subset\;U\cap\{an+b\}.$$

Consequently, $x\in\operatorname{cl}\{an+b\}$. □

Theorem 3.6.
Let $a=p_1^{\alpha_1}\dots p_k^{\alpha_k}$ be the prime power factorization of a. Define

$$A=\left\{l\le a:(p_i,l)=1\text{ or } l\equiv b\left(\operatorname{mod}p_i^{\alpha_i}\right)\text{ for each } i\in\{1,\dots,k\}\right\}.$$

Then $\operatorname{cl}\{an+b\}=\bigcup_{l\in A}\{an+l\}$. In particular, if a is square-free and the arithmetic progression $\{an+b\}$ is open, then $\operatorname{cl}\{an+b\}=\mathbb{N}$.

Proof. First assume $x\in\operatorname{cl}\{an+b\}$. By Theorems 3.5 and 3.4, respectively,

$$\operatorname{cl}\{an+b\}=\bigcap_{i=1}^{k}\operatorname{cl}\{p_i^{\alpha_i}n+b\}=\bigcap_{i=1}^{k}\left(\{p_i^{\alpha_i}n+b_i\}\cup(\mathbb{N}\setminus\{p_in\})\right),$$

where $b_i\equiv b\,(\operatorname{mod}p_i^{\alpha_i})$ and $b_i\le p_i^{\alpha_i}$ for each $i\in\{1,\dots,k\}$. We consider three cases.

Case 1: $x\in\bigcap_{i=1}^{k}\{p_i^{\alpha_i}n+b_i\}$. By CRT, there is exactly one $l\in\mathbb{N}$ such that $1\le l\le p_1^{\alpha_1}\dots p_k^{\alpha_k}$ and

$$\bigcap_{i=1}^{k}\{p_i^{\alpha_i}n+b_i\}=\left\{\left(p_1^{\alpha_1}\dots p_k^{\alpha_k}\right)n+l\right\}=\{an+l\}.$$

Since $b_i\equiv b\left(\operatorname{mod}p_i^{\alpha_i}\right)$ and $b_i\le p_i^{\alpha_i}$ for each $i\in\{1,\dots,k\}$, we have $\left\{p_i^{\alpha_i}n+b\right\}\subset\left\{p_i^{\alpha_i}n+b_i\right\}$ for each $i\in\{1,\dots,k\}$. Hence

$$\{an+b\}=\bigcap_{i=1}^{k}\{p_i^{\alpha_i}n+b\}\subset\bigcap_{i=1}^{k}\{p_i^{\alpha_i}n+b_i\}=\{an+l\},$$

which proves $l\equiv b\,(\operatorname{mod}a)$. Consequently, $l\equiv b\left(\operatorname{mod}p_i^{\alpha_i}\right)$ for each $i\in\{1,\dots,k\}$. Since $l\le a$, we obtain that $l\in A$, whence $x\in\bigcup_{l\in A}\{an+l\}$.

Case 2: $x\in\bigcap_{i=1}^{k}(\mathbb{N}\setminus\{p_in\})$. Observe

$$\bigcap_{i=1}^{k}(\mathbb{N}\setminus\{p_in\})=\bigcap_{i=1}^{k}\bigcup_{d=1}^{p_i-1}\bigcup_{t=0}^{p_i^{\alpha_i-1}-1}\{p_i^{\alpha_i}n+(p_it+d)\}.$$

So, for each $i \in \{1, \dots, k\}$ there are $d_i \in \{1, \dots, p_i - 1\}$ and $t_i \in \{0, \dots, p^{\alpha_i - 1} - 1\}$ such that $x \in \{p_i^{\alpha_i} n + (p_i t_i + d_i)\}$. This implies $x \in \bigcap_{i=1}^{k} \{p_i^{\alpha_i} n + (p_i t_i + d_i)\}$. By CRT, there is exactly one $l \in \mathbb{N}$ such that $1 \le l \le p_1^{\alpha_1} \dots p_k^{\alpha_k}$ and

$$\bigcap_{i=1}^{k} \{p_i^{\alpha_i} n + (p_i t_i + d_i)\} = \{(p_1^{\alpha_1} \dots p_k^{\alpha_k}) n + l\} = \{an + l\}.$$

Moreover, since $d_i < p_i$ for each $i \in \{1, \dots, k\}$, we have $(d_i, p_i) = 1$ for each $i \in \{1, \dots, k\}$. Therefore, $(p_i t_i + d_i, p_i) = 1$ for each $i \in \{1, \dots, k\}$ and finally, $(l, p_i) = 1$ for each $i \in \{1, \dots, k\}$. This proves that $l \in A$, whence $x \in \bigcup_{l \in A} \{an + l\}$.

Case 3: There are a number $r \in \{1, \dots, k-1\}$ and a permutation $\{\sigma_1, \dots, \sigma_k\}$ of the set $\{1, \dots, k\}$ such that $x \in \bigcap_{i=1}^{r} \{p_{\sigma_i}^{\alpha_{\sigma_i}} n + b_{\sigma_i}\} \cap \bigcap_{i=r+1}^{k} (\mathbb{N} \setminus \{p_{\sigma_i} n\})$. By CRT, there is exactly one $s \in \mathbb{N}$ such that $1 \le s \le p_{\sigma_1}^{\alpha_{\sigma_1}} \dots p_{\sigma_r}^{\alpha_{\sigma_r}}$ and $\bigcap_{i=1}^{r} \{p_{\sigma_i}^{\alpha_{\sigma_i}} n + b_{\sigma_i}\} = \{(p_{\sigma_1}^{\alpha_{\sigma_1}} \dots p_{\sigma_r}^{\alpha_{\sigma_r}}) n + s\}$. Moreover,

$$\bigcap_{i=r+1}^{k} (\mathbb{N} \setminus \{p_{\sigma_i} n\}) = \bigcap_{i=r+1}^{k} \bigcup_{d=1}^{p_{\sigma_i} - 1} \bigcup_{t=0}^{p_{\sigma_i}^{\alpha_{\sigma_i} - 1} - 1} \{p_{\sigma_i}^{\alpha_{\sigma_i}} n + (p_{\sigma_i} t + d)\}.$$

So, for each $i \in \{r+1, \dots, k\}$ there are $d_i \in \{1, \dots, p_{\sigma_i} - 1\}$ and $t_i \in \{0, \dots, p^{\alpha_{\sigma_i} - 1} - 1\}$ such that $x \in \{p_{\sigma_i}^{\alpha_{\sigma_i}} n + (p_{\sigma_i} t_i + d_i)\}$. Therefore,

$$x \in \{(p_{\sigma_1}^{\alpha_{\sigma_1}} \dots p_{\sigma_r}^{\alpha_{\sigma_r}}) n + s\} \cap \bigcap_{i=r+1}^{k} \{p_{\sigma_i}^{\alpha_{\sigma_i}} n + (p_{\sigma_i} t_i + d_i)\}.$$

By CRT, there is exactly one $z \in \mathbb{N}$ such that $1 \le z \le p_{\sigma_{r+1}}^{\alpha_{\sigma_{r+1}}} \dots p_{\sigma_k}^{\alpha_{\sigma_k}}$ and

$$x \in \{(p_{\sigma_1}^{\alpha_{\sigma_1}} \dots p_{\sigma_r}^{\alpha_{\sigma_r}}) n + s\} \cap \{(p_{\sigma_{r+1}}^{\alpha_{\sigma_{r+1}}} \dots p_{\sigma_k}^{\alpha_{\sigma_k}}) n + z\}.$$

Now, using once more CRT we obtain that there is exactly one positive integer $l \le a$ such that $x \in \{an + l\}$. Additionally, since $(d_i, p_{\sigma_i}) = 1$ for each $i \in \{r+1, \dots, k\}$, we have $(p_{\sigma_i} t_i + d_i, p_{\sigma_i}) = 1$ for each $i \in \{r+1, \dots, k\}$ and finally, $(p_{\sigma_{r+1}} \dots p_{\sigma_k}, z) = 1$. So, it is easy to see that

$$l \equiv s \pmod{p_{\sigma_1}^{\alpha_{\sigma_1}} \dots p_{\sigma_r}^{\alpha_{\sigma_r}}} \qquad \text{and} \qquad (p_{\sigma_{r+1}} \dots p_{\sigma_k}, l) = 1. \tag{2}$$

Since $b_{\sigma_i} \equiv b \pmod{p_{\sigma_i}^{\alpha_{\sigma_i}}}$ and $b_{\sigma_i} \le p_{\sigma_i}^{\alpha_{\sigma_i}}$ for each $i \in \{1, \dots, r\}$, we have $\{p_{\sigma_i}^{\alpha_{\sigma_i}} n + b\} \subset \{p_{\sigma_i}^{\alpha_{\sigma_i}} n + b_{\sigma_i}\}$ for each $i \in \{1, \dots, r\}$. Hence

$$\{(p_{\sigma_1}^{\alpha_{\sigma_1}} \dots p_{\sigma_r}^{\alpha_{\sigma_r}}) n + b\} = \bigcap_{i=1}^{r} \{p_{\sigma_i}^{\alpha_{\sigma_i}} n + b\} \subset \bigcap_{i=1}^{r} \{p_{\sigma_i}^{\alpha_{\sigma_i}} n + b_{\sigma_i}\} = \{(p_{\sigma_1}^{\alpha_{\sigma_1}} \dots p_{\sigma_r}^{\alpha_{\sigma_r}}) n + s\},$$

which implies

$$s \equiv b \pmod{p_{\sigma_1}^{\alpha_{\sigma_1}} \dots p_{\sigma_r}^{\alpha_{\sigma_r}}}. \tag{3}$$

By conditions (2) and (3), $l \equiv b \pmod{p_{\sigma_1}^{\alpha_{\sigma_1}} \dots p_{\sigma_r}^{\alpha_{\sigma_r}}}$, whence $l \equiv b \pmod{p_{\sigma_i}^{\alpha_{\sigma_i}}}$ for each $i \in \{1, \dots, r\}$. Moreover, by (2), $(p_{\sigma_i}, l) = 1$ for each $i \in \{r+1, \dots, k\}$. Consequently, $l \in A$, whence $x \in \bigcup_{l \in A} \{an + l\}$. This completes the first part of the proof.

Now we will show the opposite inclusion. Assume $x \in \bigcup_{l \in A} \{an + l\}$. Then $x \in \{an + l\}$ for some $l \in A$. Observe

$$\{an + l\} = \{(p_1^{\alpha_1} \dots p_k^{\alpha_k}) n + l\} = \bigcap_{i=1}^{k} \{p_i^{\alpha_i} n + l\}. \tag{4}$$

We will show

$$\left\{p_i^{\alpha_i}n+l\right\}\subset \operatorname{cl}\left\{p_i^{\alpha_i}n+b\right\} \qquad \text{for each} \quad i\in\{1,\dots,k\}. \tag{5}$$

Fix $i\in\{1,\dots,k\}$. Condition $l\in A$ implies $(p_i,l)=1$ or $l\equiv b\left(\operatorname{mod} p_i^{\alpha_i}\right)$. If $(p_i,l)=1$, then $\left\{p_i^{\alpha_i}n+l\right\}\subset\mathbb{N}\setminus\{p_in\}$. By Theorem 3.4, $\mathbb{N}\setminus\{p_in\}\subset\operatorname{cl}\left\{p_i^{\alpha_i}n+b\right\}$, which proves that $\left\{p_i^{\alpha_i}n+l\right\}\subset\operatorname{cl}\left\{p_i^{\alpha_i}n+b\right\}$. If $l\equiv b\left(\operatorname{mod} p_i^{\alpha_i}\right)$, then by Lemma 3.2, $\operatorname{cl}\left\{p_i^{\alpha_i}n+b\right\}=\operatorname{cl}\left\{p_i^{\alpha_i}n+l\right\}$. Therefore, $\left\{p_i^{\alpha_i}n+l\right\}\subset\operatorname{cl}\left\{p_i^{\alpha_i}n+b\right\}$, which completes the proof of (5). So, using (4), (5), and Theorem 3.5 we obtain

$$x\in\{an+l\}=\bigcap_{i=1}^{k}\left\{p_i^{\alpha_i}n+l\right\}\subset\bigcap_{i=1}^{k}\operatorname{cl}\left\{p_i^{\alpha_i}n+b\right\}=\operatorname{cl}\{an+b\}.$$

Finally, observe that if a is square-free, then $a=p_1\dots p_k$. Since

$$\{an+b\}=\bigcap_{i=1}^{k}\{p_in+b\}$$

and $\{an+b\}$ is open, $\{p_in+b\}$ is also open for each $i\in\{1,\dots,k\}$. So, Theorem 3.5 and condition (iii) of Theorem 3.4 imply

$$\operatorname{cl}\{an+b\}=\bigcap_{i=1}^{k}\operatorname{cl}\{p_in+b\}=\mathbb{N}.$$

This completes the proof. □

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